

DISCRIMINANTS IN THE GROTHENDIECK RING

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ABSTRACT. We consider the “limiting behavior” of *discriminants*, by which we mean informally the locus in some parameter space of some type of object where the objects have certain singularities. We focus on the space of partially labeled points on a variety X , and linear systems on X . These are connected — we use the first to understand the second. We describe their classes in the Grothendieck ring of varieties, as the number of points gets large, or as the line bundle gets very positive. They stabilize in an appropriate sense, and their stabilization is given in terms of motivic zeta values. Motivated by our results, we conjecture that the symmetric powers of geometrically irreducible varieties stabilize in the Grothendieck ring (in an appropriate sense). Our results extend parallel results in both arithmetic and topology. We give a number of reasons for considering these questions, and propose a number of new conjectures, both arithmetic and topological.

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1. INTRODUCTION

We study the classes of discriminants (loci in a moduli space of objects with specified singularities) and their complements in the Grothendieck ring of varieties, focusing on the cases of moduli of hypersurfaces and configuration spaces of points. The main contributions of this paper are two theorems (Theorems 1.13 and 1.30) and one conjecture (“motivic stabilization of symmetric powers”, Conjecture 1.25).

I. (Theorem 1.13, the limiting motive of the space of hypersurfaces with a given number of singularities, §3) If \mathcal{L} is an ample line bundle on a smooth variety X , we show that the motive

of the subset of the linear system $|\mathcal{L}^{\otimes j}|$ consisting of divisors with precisely s singularities (normalized by $|\mathcal{L}^{\otimes j}|$), tends to a limit as $j \rightarrow \infty$ (in the completion of the localization of the Grothendieck ring at $\mathbf{L} := [\mathbb{A}^1]$), given explicitly in terms of the motivic zeta function of X .

II. (*Conjecture 1.25, motivic stabilization of symmetric powers, §4*) We conjecture that if X is geometrically irreducible, then the ratio $[\mathrm{Sym}^n X]/\mathbf{L}^{n \dim X}$ tends to a limit. This is an algebraic version of the Dold-Thom theorem, and is also motivated by the Weil conjectures. We give a number of reasons for considering this conjecture.

III. (*Theorem 1.30, the limiting motive of discriminants in configuration spaces, §5*) We show that if X is geometrically irreducible and satisfies motivic stabilization (**II**, e.g. if X is stably rational, see Motivation 1.26(i)), then the motive of strata (and their closure) of configurations of points with given “discriminant” (clumping of points) tends to a limit as the number of points $n \rightarrow \infty$, and (more important) we describe the limit in terms of motivic zeta values. In the case of s multiple points, the result is the same as that of **I**, except the expression in terms of motivic zeta functions is evaluated at a different value (see Theorem 1.39 and (1.40)). The reliance on the motivic stabilization conjecture can be removed by specializing to Hodge structures, where the analogous conjecture holds (see Motivation 1.26(ii)), or by working with generating series (Theorem 5.2).

These results are motivated by a number of results in number theory and topology (including, notably, stability/stabilization theorems), and they generalize analogues of many of these statements. (An elementary motivation is an analogue of both **I** and **III** for $X = \mathrm{Spec} \mathbb{Z}$: the probability of an integer being square free is $1/\zeta(2)$. One has to first make sense of the word “probability” as a limit, then show that the limit is a zeta value. These features will be visible in our arguments as well.) Our results also support Denef and Loeser’s motto [DL3, I. 1-2]: “rational generating series occurring in arithmetic geometry are motivic in nature”.

Our results suggest a number of new conjectures in arithmetic, algebraic geometry, and topology that may be tractable by other means. We label these smaller new conjectures by letters A through H. The reason for stating many of these conjectures is not necessarily an expectation that they will be true, but because either a proof or a counterexample should provide significant new insight. Hence throughout this paper, *conjecture* should perhaps be interpreted as *conjecture/speculation/question*.

We now describe these results in more detail and context. We first set notation and review background about the Grothendieck ring. Thereafter, the discussion of **I**, **II**, and **III** can be read largely in any order.

1.1. Notation and background: The Grothendieck ring of varieties. Throughout, \mathbb{K} is a field. A *variety* is a reduced separated finite type \mathbb{K} -scheme; X will usually denote a variety, and d will be its dimension.

The Grothendieck ring of varieties $\mathcal{M} := K_0(\text{Var}_{\mathbb{K}})$ is defined as follows. As an abelian group, it is generated by the classes of finite type \mathbb{K} -schemes up to isomorphism. The class of a scheme X in \mathcal{M} is denoted $[X]$, but we often drop the brackets for convenience. The group relations are generated by the following: if Y is a closed subscheme of X , and U is its (open) complement, then $[X] = [U] + [Y]$. In particular, taking $Y = X^{\text{red}}$, we have $[X] = [X^{\text{red}}]$, so nilpotents play no role in our discussions. The product $[X][Y] := [X \times_{\mathbb{K}} Y]$ makes \mathcal{M} into a commutative ring, with $[\text{Spec } \mathbb{K}]$ as unit.

Any morphism ϕ from \mathcal{M} to another ring is called a *motivic measure*. Here are two important examples. (i) If $\mathbb{K} = \mathbb{F}_q$, there is a *point counting map* $\# : \mathcal{M} \rightarrow \mathbb{Z}$.

(ii) For convenience, we call the Grothendieck group of mixed Hodge structures *virtual Hodge structures*. This is the same as the sum (over all weights) of the Grothendieck group of \mathbb{Q} -Hodge structures. If $\mathbb{K} = \mathbb{C}$, there is a map from \mathcal{M} to the group of virtual Hodge structures, defined by taking each variety X to $\sum_k (-1)^k [H_c^k(X, \mathbb{Q})]$. This descends to a motivic measure HS from \mathcal{M} to the group of virtual Hodge structures: given complementary subsets $Y, U \subset X$ as in two paragraphs previous, the long exact sequence for cohomology with compact support respects mixed Hodge structure. This motivic measure specializes further to the *Hodge-Deligne polynomial* $e : \mathcal{M} \rightarrow \mathbb{Z}[x, y]$, where for a variety X ,

$$e(X) = \sum_k (-1)^k h^{p,q}(H_c^k(X)) x^p y^q.$$

(This has also been called the E-polynomial, the virtual Hodge polynomial, the Serre polynomial, and the Hodge-Euler polynomial.) If X is smooth and proper, $e(X)$ determines the Hodge numbers on each of the cohomology groups, and in particular the Betti numbers $h^i(X)$.

1.2. Principle: Occam's razor for Hodge structures. We point out a well-known (if vague) principle. For a variety X that is not smooth and proper, the virtual Hodge structure does not determine the Hodge structures on each $H_c^k(X)$ (and similarly the Hodge-Deligne polynomial does not determine the Hodge numbers $h^{p,q}(H_c^k(X))$) because the contributions from different k are mixed. But in many cases there is a simplest Hodge structure on all the H_c^k 's compatible with the virtual Hodge structure, and it is reasonable to wonder if the Hodge structure on the H_c^k 's is this simplest possibility in some simple cases. Similarly, if the virtual Hodge structures stabilize in some sense in some family of examples, one should expect that this arises because the actual Hodge structures on the H_c^k 's (and hence the “compact type” Betti numbers) also stabilize.

1.3. Inverting L . We denote the “Lefschetz motive” $[\mathbb{A}^1]$ by L . There are many reasons to consider the localization \mathcal{M}_L (including motivic integration; possible rationality of the motivic zeta function, §1.8; and the desirability of a homotopy axiom). This paper suggests additional reasons. The motivic measures HS and $\#$ clearly extend to \mathcal{M}_L .

1.4. *Completion with respect to the dimensional filtration.* The Grothendieck ring \mathcal{M} is filtered by the subgroups generated by varieties of dimension at most d , as d varies. This *dimensional filtration* clearly extends to $\mathcal{M}_{\mathbf{L}}$. Let $\widehat{\mathcal{M}}_{\mathbf{L}}$ be the completion of $\mathcal{M}_{\mathbf{L}}$ with respect to the dimensional filtration. As explained in [B, §1.5], $\widehat{\mathcal{M}}_{\mathbf{L}}$ inherits not just a group structure, but also a ring structure. The ring $\widehat{\mathcal{M}}_{\mathbf{L}}$ was originally introduced by Kontsevich [Kon] as the ring where values of motivic integrals lie, in his theory of motivic integration (see [DL1, DL2, Lo]). Note that the motivic measure HS extends to this completion, after suitably extending the codomain. But the point-counting motivic measure does not — the point counting map $\# : \mathcal{M}_{\mathbf{L}} \rightarrow \mathbb{Q}$ is not continuous (consider the sequence $(2q)^n \mathbf{L}^{-n}$).

1.5. *Observation.* The symmetric product $\mathrm{Sym}^n X$ will be central to us. If X is not quasiprojective, then $\mathrm{Sym}^n X$ might not be a variety. However, $[\mathrm{Sym}^n X]$ may be interpreted as an element of $\widehat{\mathcal{M}}_{\mathbf{L}}$, by [E, Thm. 1.2] (note that $\mathrm{Sym}^n X$ is represented by an algebraic space), which will suffice for our purposes; we will hereafter pass over this technical point without comment. We note for future reference that (i) if X is rational, then $[\mathrm{Sym}^n X]$ is invertible in $\widehat{\mathcal{M}}_{\mathbf{L}}$, and (ii) if $\mathbb{K} = \mathbb{C}$ then $\mathrm{HS}(\mathrm{Sym}^n X)$ is invertible in $\mathrm{HS}(\widehat{\mathcal{M}}_{\mathbf{L}})$ (for any X).

1.6. *The motivic zeta function.* Let $\mathbf{Z}_X(t) := \sum_{n \geq 0} [\mathrm{Sym}^n X] t^n \in \mathcal{M}[[t]]$ be the *motivic zeta function* (defined by Kapranov, [Kap, (1.3)]). If $\mathbb{K} = \mathbb{F}_q$, then $\#$ sends $\mathbf{Z}_X(t)$ to the Weil zeta function $\zeta_X(s)$, where $t = q^{-s}$. Motivated by this, we define (for *any* \mathbb{K}) $\zeta_X(\mathfrak{m}) := \mathbf{Z}_X(\mathbf{L}^{-\mathfrak{m}})$. We use bold fonts for both ζ and \mathbf{Z} in order to distinguish them from the Weil zeta function(s).

Let $\mathrm{Sym}_{[s]}^n X \subset \mathrm{Sym}^n X$ (not to be confused with $\mathrm{Sym}_s^n X \subset \mathrm{Sym}^n X$, to be defined in §1.38) be the locally closed subset of $\mathrm{Sym}^n X$ consisting of unordered n -tuples of points supported on exactly s (distinct, geometric) points. Let $\mathbf{Z}_X^{[s]}(t) = \sum_n [\mathrm{Sym}_{[s]}^n X] t^n$, so $\mathbf{Z}_X(t) = \mathbf{Z}_X^{[0]}(t) + \mathbf{Z}_X^{[1]}(t) + \mathbf{Z}_X^{[2]}(t) + \dots$. It is straightforward to write $\mathbf{Z}_X^{[s]}(t)$ in terms of $\mathrm{Sym}^n X$'s and rational functions of t . For example,

$$(1.7) \quad \mathbf{Z}_X^{[0]}(t) = 1, \quad \mathbf{Z}_X^{[1]}(t) = \frac{t}{1-t} X, \quad \text{and} \\ \mathbf{Z}_X^{[2]}(t) = \frac{t^2}{1-t^2} \mathrm{Sym}^2 X + \frac{t^3}{(1-t^2)(1-t)} X^2 - \frac{t^2}{(1-t^2)(1-t)} X.$$

Define $\zeta_X^{[s]}(\mathfrak{m}) := \mathbf{Z}_X^{[s]}(\mathbf{L}^{-\mathfrak{m}})$.

1.8. *Rationality of the motivic zeta function.* Kapranov [Kap, Rem. 1.3.5] asked if the motivic zeta function is rational, given that its specialization $\zeta_X(t)$ is rational. (This question is related to **II**, see §1.25(iv).) Further evidence is that the motivic measure to Hodge structures $\mathrm{HS}(\mathbf{Z}_X(t))$ is rational, which was shown by Cheah (actually predating the definition of the motivic zeta function). We note that some care is necessary to say what is meant by rationality, see [LL2, §2].

1.9. Theorem (Cheah, [C1], see also [C2]). — Suppose X is a complex variety. Then

$$\mathrm{HS}(\mathbf{Z}_X(t)) = \prod_{i=0}^{\infty} \left((1-t)^{h^i(X)} \right)^{(-1)^{i+1}}$$

where if V is a mixed Hodge structure, then $(1-t)^{[V]}$ is interpreted as $\sum_{j=0}^{\infty} (-1)^j [\wedge^j V] t^j$, and $[\cdot]$ indicates the class in virtual Hodge structures.

Cheah's argument deals only with the specialization to the Hodge-Deligne polynomials (actually an enrichment of this), but the proof can be adapted to establish Theorem 1.9.

1.10. However, Larson and Lunts [LL1, LL2] showed that $\mathbf{Z}_X(t)$ is *not* always rational in $\mathcal{M}[[t]]$. But an important (if vague) question remains: where between \mathcal{M} and the special motivic measures of point-counting or Hodge structures is the motivic zeta function rational? In particular, the argument of Larson and Lunts does not apply to $\mathcal{M}_{\mathbf{L}}$.

1.11. Conjecture [DL3, Conj. 7.5.1]. — The generating series $\mathbf{Z}_X(t)$ is rational in $\mathcal{M}_{\mathbf{L}}[[t]]$.

1.12. Moduli of Hypersurfaces.

Our main result on hypersurfaces is the following, proven in Section 3.

1.13. Theorem. — Let X be a smooth projective variety of pure dimension $d > 0$ with an ample line bundle \mathcal{L} . Let $H^0(X, \mathcal{L}^{\otimes j})^s$ be the constructible subset of $H^0(X, \mathcal{L}^{\otimes j})$ corresponding to divisors on X with exactly s singular geometric points. Then

$$(1.14) \quad \lim_{j \rightarrow \infty} \frac{[H^0(X, \mathcal{L}^{\otimes j})^s]}{[H^0(X, \mathcal{L}^{\otimes j})]} = \frac{\zeta_X^{[s]}(d+1)}{\zeta_X(d+1)} \quad (\text{in } \widehat{\mathcal{M}_{\mathbf{L}}}).$$

For example (using (1.7)), the motivic (limiting) probability of a divisor being smooth (i.e. $s = 0$) is

$$1/\zeta_X(d+1),$$

and the motivic (limiting) probability of a divisor having precisely one singularity ($s = 1$) is

$$\frac{X/\mathbf{L}^{-(d+1)}}{1 - 1/\mathbf{L}^{-(d+1)}} \cdot \frac{1}{\zeta_X(d+1)}.$$

1.15. Remarks.

- (i) Note that the limiting motivic density (the right side of (1.14)) is independent of \mathcal{L} .
- (ii) Even to establish the result for $\mathbb{K} = \mathbb{C}$, the argument requires the use of finite fields (see Lemma 3.19).
- (iii) The hypothesis of projectivity can be weakened to quasiprojectivity by taking appropriate care in defining the motivic probability (cf. §1.16). We leave this variation to the interested reader (see Motivation 1.16).

(iv) If $s > 1$, $H^0(X, \mathcal{L}^{\otimes j})^s$ will in general not be locally closed. For example, consider $s = 2$ and $d = 2$, in the neighborhood of a curve with a tacnode and a node.

(v) If instead of s singular points, we require s singular geometric points that are A_1 -singularities, then the corresponding locus is locally closed (not just constructible), and the limit again exists, and can be interpreted as follows. Let \mathbf{P}_d be the motivic probability that a singular point p of a divisor on a smooth d -fold X passing through p is an A_1 -singularity. (This can be made precise in the obvious way, by considering $\mathrm{Sym}^2(\Omega_X|_p)$.) Then multiply the right side of (1.14) by $(\mathbf{P}_d)^s$. We omit the justification. However, see Conjecture 1.20 below.

(vi) A simple extension of the argument yields the following result: the motivic (limiting) probability that a section of $\mathcal{L}^{\otimes j}$ has no m -multiple points is $1/\zeta_X\left(\binom{n+m-1}{n}\right)$. (An m -multiple point is a singular point of multiplicity at least m — the defining equation vanishes to order at least m .)

(vii) A variation of the proof of Theorem 1.13 (see Remark 3.25) yields the following. Let $H^0(X, \mathcal{L}^{\otimes j})^{s \text{ ordered}}$ be the space of sections of $H^0(X, \mathcal{L}^{\otimes j})$ along with a choice of s ordered (disjoint) singular (geometric) points. Then

$$\lim_{j \rightarrow \infty} \frac{[H^0(X, \mathcal{L}^{\otimes j})^{s \text{ ordered}}]}{[H^0(X, \mathcal{L}^{\otimes j})]} = \frac{[X^s \setminus \Delta]}{\zeta_X(d+1)} \left(\frac{1/L^{d+1}}{1 - 1/L^{d+1}} \right)^s \quad (\text{in } \widehat{\mathcal{M}}_L)$$

where $\Delta \subset X^s$ is the “big diagonal”.

We give three motivations for Theorem 1.13.

1.16. First Motivation: Poonen’s probability for a hypersurface to be smooth. Poonen’s “Bertini Theorem over finite fields” [P2, Thm. 1.1] is (informally) the following. Suppose $X \subset \mathbb{P}^N$ is a smooth projective variety over \mathbb{F}_q of dimension d . (Poonen states his result more generally in the quasiprojective case, and ours can be so extended as well, see Remark 1.15(iii).) As the base field is finite, one can make sense of the probability p_j of a hypersurface of degree j (defined over \mathbb{F}_q) intersecting X along a smooth (codimension 1) subvariety. Poonen shows that $\lim_{j \rightarrow \infty} p_j$ exists, and equals $1/\zeta_X(d+1)$, where ζ_X is the Weil zeta function.

Theorem 1.13 in the case $s = 0$ is the motivic analogue of Poonen’s result. However, this case of Theorem 1.13 neither implies nor is implied by [P2, Thm. 1.1]; the limits taken in both cases are not compatible, because the dimensional filtration has no relation to point-counting (§1.4). Furthermore, the methods of proof are unrelated (except at a very superficial level). Based on Theorem 1.13, it is reasonable to conjecture the following.

1.17. Conjecture A. — Suppose $X \subset \mathbb{P}^N$ is a smooth, dimension d , variety over \mathbb{F}_q . Let $p_j^{[s]}$ be the probability of a hypersurface of degree j (defined over \mathbb{F}_q) intersecting X along a subvariety

with exactly s non-smooth geometric points. Then

$$\lim_{j \rightarrow \infty} p_j^{[s]} = \frac{\zeta_X^{[s]}(d+1)}{\zeta_X(d+1)},$$

where $\zeta_X^{[s]}$ is defined analogously to $\zeta_X^{[s]}$ (§1.6).

The case of \mathbb{P}^1 follows from Theorem 1.39(b), and the heuristic given after the statement of Theorem 1.39 suggests this result for smooth curves ($d = 1$) in general.

1.18. *Second Motivation: motives of Severi varieties.* Severi varieties are closed subsets of linear systems on a smooth projective surface (X, \mathcal{L}) with a fixed number of singularities. Göttsche's conjecture (now a theorem, see [Tz] and [KST]) states that the degrees of these varieties have strong structure: if the line bundle is sufficiently ample, then the degree can be read off from a universal formula involving four universal generating series (two of them quasimodular forms), and the four numerical invariants of (X, \mathcal{L}) . (See [KST] for a more precise statement.)

Theorem 1.13 states that not only does the degree of the Severi variety have a strong structure (related to modular forms), the motive does as well (related to zeta functions). (If one wishes to restrict to nodal curves, i.e. A_1 -singularities, as is usually the case for Severi varieties, see Remark 1.15(v). We conjecture similar structure if one considers more general singularity types, in vague analogy with [LT], extending the proof [Tz] of Göttsche's conjecture, or [R], extending the proof [KST].)

1.19. *Third Motivation: Vassiliev's work on topology of discriminants and their complements.* Vassiliev's fundamental work on topology of discriminants or discriminant complements (in a broad sense, encompassing different singularity types) can be seen as philosophical and direct motivation for this work. We left it for last as it leads to further questions and conjectures. We briefly summarize some of Vassiliev's work; see [V2] and the references (by Vassiliev) therein for more. Vassiliev determines the cohomology ring of the space of holomorphic functions in d complex variables giving a smooth divisor, [V1, Thm. 1]. His argument works without change to apply to *algebraic* (i.e. polynomial) functions in d complex variables. This predicts the same Betti numbers as the application of Occam's Razor 1.2 to Theorem 1.13 in the special case $X = \mathbb{C}^d$, $\mathcal{L} = \mathcal{O}$ (see Remark 1.15(iii)), which suggests that if S is the space of algebraic functions with smooth divisor, $h_i(S) = 0$ for $i \neq 0, 1$, and $h_0(S) = h_1(S) = 1$.

Thus motivated, we conjecture the following.

1.20. **Conjecture B.** — Suppose X is a smooth complex variety, and let \mathcal{L} be an ample line bundle on X . Let X_j^s be the space of sections of $\mathcal{L}^{\otimes j}$ that vanish on a divisor of X singular at precisely s points, each of which is an A_1 -singularity (so X_j^0 is the space of sections whose vanishing scheme is a smooth divisor of X). Let Y_j^1 be the space of sections that vanish on a divisor of X singular

at one point. Then the rational homology type of Y_j^1 and each X_j^s stabilizes (i.e. for every i , we have that the singular homology groups $h_i(Y_j^1)$ and $h_i(X_j^s)$ stabilize for $j \gg_i 0$), and each limit is independent of \mathcal{L} .

See Remarks 1.15(iv) and (v) for motivation. The case most of interest is $s = 0$, and even when X is a projective or affine variety this case is not clear. (One might hope that Vassiliev's arguments can be extended to this case. Note in particular that his constructions can be algebraized, and that the spectral sequence used in his proof degenerates at E^1 , see [V2, p. 212].)

Also, even in the case of $X = \mathbb{C}^d$, this conjecture is not clear (except for $s = 0$, which is Vassiliev's result). One can also give an analogous form of Conjecture B for divisors of X with no singularity of multiplicity m (or exactly one such).

It would be interesting not just to know that the limits in Conjecture B (and its variant with m -fold points) exist, but to actually describe the limit rational homology type (i.e., the Poincare series), and in particular, compare these limits to the limit motives given in Theorem 1.13. In particular, one could hope there is some imprecise dictionary between motivic zeta-values of X and rational homology types built in some way out of X . Somewhat more precisely, when one sees a motivic zeta-value in a limiting formula for some geometric problem, one might expect to see the corresponding rational homology type in the stabilization.

1.21. Vague question. — *What rational homology type does $1/\zeta_X(N)$ correspond to?*

This vague question suggests explicit questions. For example, in [T, p. 1066 (2)], Totaro gives two simple complex projective manifolds, $X = \mathbb{P}^1 \times \mathbb{P}^2$ and $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$, with $[X] = [Y]$ (and so X and Y have the same Betti numbers), such that the space of 3 distinct points on X has different Betti numbers than the space of 3 distinct points on Y . In light of Theorem 1.30(a) below (with $\lambda = \emptyset$), Vague Question 1.21 leads one to ask if this pathology goes away as the number of points gets large. More precisely, does the cohomology of the space of n distinct points on X agree with that of n distinct points on Y in any particular degree as $n \rightarrow \infty$? (Totaro pointed out to us that [FT] has a different description of the Betti numbers of configuration spaces that may be better to approach this problem.)

We also conjecture that when the Hodge-Deligne series of the limiting motive is finite, then the limit of the corresponding Poincare series is finite. As an important explicit example, we have the following.

1.22. Conjecture C. — *For a smooth, complex variety X , if $e(\zeta_X(d+1))$ is a polynomial in x^{-1}, y^{-1} (e.g. if X is projective with no odd degree cohomology), then for i sufficiently large, and j sufficient large depending on i , we have $h_i(X_j^0) = 0$.*

Under the hypotheses of Conjecture C, it is natural to wonder whether an application of Occam's Razor 1.2 to the limit motive gives the correct limit Betti numbers. For $X = \mathbb{A}^d$, the work of Vassiliev mentioned in §1.19 implies that the answer is yes. For \mathbb{P}^1 , [Ch1, Prop. 4.5] can be used to show that the answer is no for X_j^0 , but yes for X_j^0/\mathbb{C}^* . (We thank T. Church for explaining this to us, [Ch3].) As just one open example for X_j^0/\mathbb{C}^* , we highlight the case of plane curves.

1.23. Conjecture D. — *Let X_j^0/\mathbb{C}^* be the space of degree j smooth projective plane curves (a quasiprojective manifold of complex dimension $\binom{j+2}{2} - 1$). Then $\lim_{j \rightarrow \infty} h_i(X_j^0) = 1$ for $i = 0, 3, 5, 10$ and $\lim_{j \rightarrow \infty} h_i(X_j^0) = 0$ otherwise.*

We remark that the fundamental group of X_j^0/\mathbb{C}^* (parametrizing smooth plane curves) was computed by Lönne, [Lö, Main Thm.], and also behaves well as $j \rightarrow \infty$.

1.24. Motivic stabilization of symmetric powers.

1.25. Conjecture (Motivic stabilization of symmetric powers). — *Suppose X is a geometrically irreducible variety of dimension d . Then the limit $\lim_{n \rightarrow \infty} [\mathrm{Sym}^n X]/\mathbf{L}^{dn}$ exists in $\widehat{\mathcal{M}}_{\mathbf{L}}$.*

(To see the necessity of the geometric irreducibility hypothesis, consider the case where X is two points, or see Motivation 1.26 below.) If Conjecture 1.25 holds for X , we say that MSSP (or motivic stabilization of symmetric powers) holds for X . If $\phi : \widehat{\mathcal{M}}_{\mathbf{L}} \rightarrow \phi(\widehat{\mathcal{M}}_{\mathbf{L}})$ is a continuous ring homomorphism (extending a continuous motivic measure $\phi : \mathcal{M}_{\mathcal{L}} \rightarrow \phi(\widehat{\mathcal{M}}_{\mathbf{L}})$), we say MSSP_{ϕ} holds for X if

$$\mathrm{SP}_{\phi}(X) := \lim_{n \rightarrow \infty} \frac{[\mathrm{Sym}^n X]}{\mathbf{L}^{dn}} \text{ exists in } \phi(\widehat{\mathcal{M}}_{\mathbf{L}}),$$

where we abuse notation by using $[Z]$ to denote $\phi([Z])$. Our use of the topology notation $\mathrm{SP}(X)$ for infinite symmetric product is motivated by the Dold-Thom theorem, see Motivation 1.26(v) below. (It may be suggestive to write $[\mathrm{Sym}^n X]/[\mathbf{L}^{dn}]$ as $[\mathrm{Sym}^n X]/[\mathrm{Sym}^n \mathbb{A}^{\dim X}]$, using Proposition 4.1.)

1.26. Motivation. We give a number of motivations for considering Conjecture 1.25.

(i) We show that Conjecture 1.25 holds (or fails) on stable birational equivalence classes (Proposition 4.1 combined with Proposition 4.2), and for curves with a rational point (Proposition 4.5). In particular, as Conjecture 1.25 clearly holds for a point, it holds for all stably rational varieties.

(ii) Conjecture 1.25 is true upon specialization to Hodge structures (i.e., $\mathrm{MSSP}_{\mathrm{HS}}$ holds, for all X), which can be shown from Theorem 1.9.

(iii) The analogue for point-counting also holds. (Unlike (ii), this is only an analogy: point-counting is not compatible with the completion with respect to the dimensional filtration, see §1.4.) More precisely, if X is a geometrically irreducible variety over a finite

field, then $\lim_{n \rightarrow \infty} \frac{\#\mathrm{Sym}^n X}{q^{dn}}$ exists. This is because, by the Weil conjectures, the generating function for the Weil zeta function $\zeta_X(t)$ has as its denominator a polynomial whose smallest root is $1/q^d$ (corresponding to the fundamental class of X), and this root appears with multiplicity 1.

(iv) Related to Conjecture 1.11 on whether the motivic zeta function $Z_X(t)$ is rational (upon localization by L), one is led to ask whether, in a suitable sense, the denominator of $Z_X(t)$ has a unique smallest root (in the sense of dimension), L^{-d} (corresponding to the “fundamental class of X ”, in further analogy to the Weil conjectures). Suitably interpreted, this would imply Conjecture 1.25.

(v) A topological motivation is the Dold-Thom theorem [DT], and more basically that the homotopy type of $\mathrm{Sym}^n X$ has a limit $SP(X)$, where X is a topological space (see, for example, [CCMM, §2] for more discussion). If $\mathbb{K} = \mathbb{C}$, then Dold-Thom implies that $h_i(\mathrm{Sym}^n X, \mathbb{Q})$ stabilizes as $n \rightarrow \infty$. If further X is smooth, then Poincaré duality holds for $\mathrm{Sym}^n X$ (with \mathbb{Q} -coefficients), because $\mathrm{Sym}^n X$ is the coarse moduli space for the orbifold (smooth Deligne-Mumford stack) X^n/\mathfrak{S}_n . The quotient by L^{dn} in the statement of Conjecture 1.25 arises because then, by Poincaré duality, $h_c^{2nd-i}(\mathrm{Sym}^n X)$ stabilizes as $n \rightarrow \infty$, and the weight $-i$ piece of $e([\mathrm{Sym}^n X]/L^{dn})$ is the weight $2nd - i$ piece of $h_c^*(\mathrm{Sym}^n X)$.

(vi) Kimura and Vistoli have given analogous conjectures for Chow groups, notably their Weak Stabilization Conjecture [KiV, Conj. 2.6] (true for curves, [KiV, Prop. 2.9(a)]) and their Strong Stabilization Conjecture [KiV, Conj. 2.13] (true for pointed curves of genus up to 4, [KiV, Cor. 2.19], and with motivation for all curves, [KiV, Rk. 2.20]).

(vii) The statement of Conjecture 1.25 contradicts each of two well-known questions (or conjectures), Conjectures 1.27 and 1.28 below, as shown by D. Litt, [L].

1.27. Piecewise Isomorphism Conjecture (Larsen-Lunts, [LL1, Qu. 1.2]; see also [LS, Assertion 1]). — *If X and Y are varieties with $[X] = [Y]$ in $K_0(\mathrm{Var}_{\mathbb{K}})$, then we can write $X = \coprod_{i=1}^n X_i$ and $Y = \coprod_{i=1}^n Y_i$ with X_i and Y_i locally closed, and $X_i \cong Y_i$ (X and Y are “piecewise isomorphic”).*

Liu and Sebag have proved Conjecture 1.27 if \mathbb{K} is algebraically closed of characteristic 0, when X is a smooth connected projective surface [LS, Thm. 4], or when X contains only finitely many rational curves [LS, Thm. 5].

1.28. Conjecture (well-known, see for example [DL3, §3.3]). — *The element L is not a zero-divisor. Equivalently, the localization $\mathcal{M} \rightarrow \mathcal{M}_L$ is an injection.*

This is more a question than a conjecture. The real (if vague) question is: “what information, if any, is lost by localizing by L ?” (It is known that \mathcal{M} is not an integral domain, see [P1, Thm. 1], [Kol, Ex. 6], [N, Thm. 22].) In light of Larsen and Lunts’ counterexample to the rationality in general of the motivic zeta function, §1.10, Conjecture 1.28 contradicts Conjecture 1.11.

1.29. Configurations of points on varieties.

For a partition λ of n , let $w_\lambda(X)$ be the locally closed subset of $\text{Sym}^n X$ that is the locus of points which occur with multiplicities precisely λ , and let $\overline{w}_\lambda(X)$ be its closure. For example, $w_{1^n}(X)$ is the configuration space of n unordered distinct geometric points, sometimes denoted $B(X, n)$ or $\text{Conf}^n(X)$. Let $1^k\lambda$ denote the partition obtained from adding k 1's to λ .

1.30. Theorem. — *Suppose X is a geometrically irreducible variety of dimension d .*

(a) *If X satisfies MSSP_ϕ , then the limits*

$$(1.31) \quad \lim_{j \rightarrow \infty} \frac{[w_{1^j\lambda}(X)]}{\mathbf{L}^{dj}} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{[\overline{w}_{1^j\lambda}(X)]}{\mathbf{L}^{dj}}$$

exist in $\phi(\widehat{\mathcal{M}}_{\mathbf{L}})$, and have finite formulas in terms of motivic zeta values, the $[\text{Sym}^i X]$, and $\text{SP}_\phi(X)$ (defined in §1.25). If furthermore the $[\text{Sym}^j X]$ are invertible in $\phi(\widehat{\mathcal{M}}_{\mathbf{L}})$ (e.g. if X is rational or $\phi = \text{HS}$, §1.5), then

$$(1.32) \quad \lim_{j \rightarrow \infty} \frac{[w_{1^j\lambda}(X)]}{[\text{Sym}^{j+|\lambda|} X]} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{[\overline{w}_{1^j\lambda}(X)]}{[\text{Sym}^{j+|\lambda|} X]}$$

exist in $\phi(\widehat{\mathcal{M}}_{\mathbf{L}})$, and have finite formulas in terms of motivic zeta values and $[\text{Sym}^i X]$.

(b) *If $\mathbb{K} = \mathbb{F}_q$, then*

$$(1.33) \quad \lim_{j \rightarrow \infty} \frac{\#w_{1^j\lambda}(X)}{\#\text{Sym}^{j+|\lambda|} X} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{\#\overline{w}_{1^j\lambda}(X)}{\#\text{Sym}^{j+|\lambda|} X}$$

exist, and have finite formulas in terms of Weil zeta values, and derivatives of the Weil zeta function.

The second statement of (a) can be interpreted as a limiting “motivic probability” ($w_{1^j\lambda}(X)$ and $\overline{w}_{1^j\lambda}(X)$ are both subsets of $\text{Sym}^{j+|\lambda|} X$), and the statement of (b) can be interpreted as a limiting probability. Corollary 5.7 is a more precise version of Theorem 1.30. Corollary 5.7 in turn is a consequence of an analogous statement about generating functions (Theorem 5.2), with *no* motivic stabilization hypotheses, which states that $\sum_j [w_{1^j\lambda}(X)] t^j$ and $\sum_j [\overline{w}_{1^j\lambda}(X)] t^j$ have finite formulas in terms of $\mathbf{Z}_X(t)$ and $[\text{Sym}^i X]$; the formulas are essentially the same as in Theorem 1.30 above.

The finite formulas of Theorem 1.30 are given recursively in Propositions 5.11 and 5.14(b). It is not hard to show that the limits exist; the main content of Theorem 1.30 (or, rather, Corollary 5.7) is the description of the limit. These limits have particularly nice descriptions in special cases. We give some now. Rather than giving three forms of each result (corresponding to (1.31)–(1.33) in Theorem 1.30), for simplicity we just discuss the “motivic probability” versions ((1.32) in Theorem 1.30), as representative of all three versions.

1.34. Fix now a rational variety X (over \mathbb{K}) of dimension d . (The reason for assuming rationality is for simplicity, so that $[\mathrm{Sym}^j X]$ is invertible in $\widehat{\mathcal{M}}_{\mathbb{L}}$, §1.5.) The limiting motivic probability (as $j \rightarrow \infty$) that j points are distinct (i.e. $\lim_{j \rightarrow \infty} [w_{1j}(X)]/[\mathrm{Sym}^j X]$) is $\zeta_X(2d)^{-1}$. (Equivalently, the limiting motivic probability that j points are not distinct — the traditional “discriminant locus” — is $1 - \zeta_X(2d)^{-1}$.) The corresponding generating function formula is

$$\sum_j [w_{1j}(X)] t^j = Z_X(t)/Z_X(t^2)$$

(a special case of Proposition 5.9(a)), which specializes, under taking Euler characteristic with compact supports, to the well-known formula for Euler characteristic of configuration spaces

$$\sum_j \chi_c(w_{1j}(X)) t^j = (1 + t)^{\chi(X)}$$

using Macdonald’s formula $\chi_c(Z_X(t)) = (1 - t)^{-\chi(X)}$ from [Mac]. We remark that there is a large body of work, going back to Macdonald [Mac], giving generating functions for motivic or topological invariants of symmetric products (see [Z, M, C2, BL1, O, MS1, MS2]) and Hilbert schemes (see [G1, GS, C2, BL2, GZLMH1, GZLMH2, BN, NW, CMOSY]). Our formulas also extend such motivic generating functions to the generalized configuration spaces $w_{1j\lambda}(X)$ and $\overline{w}_{1j\lambda}(X)$ which are the natural strata (and their closures) of symmetric products.

1.35. We return to our examples. Generalizing §1.34, the limiting motivic probability that j unordered points have a point of multiplicity (at least) a , i.e. $\lim_{j \rightarrow \infty} [\overline{w}_{1j-a}(X)]/[\mathrm{Sym}^j X]$, is $1 - \zeta_X(ad)^{-1}$ (a consequence of Proposition 5.9 and Lemma 5.4). This is an analog of the classical arithmetic fact that the proportion of a th-power-free integers is $\zeta(a)^{-1}$.

1.36. More generally (and more subtly) there is a simple description of the limiting motivic probability that there are r points of multiplicity b or worse, i.e. $\lim_{j \rightarrow \infty} [\overline{w}_{1jb^r}]/[\mathrm{Sym}^{j+br} X]$ (a consequence of Proposition 5.19 and Lemma 5.4); its simplicity is clearest in the case where $X = \mathbb{A}^d$, in which case $[\overline{w}_{1jb^r}(\mathbb{A}^d)]/[\mathrm{Sym}^{j+br} \mathbb{A}^d] = 1/\mathbb{L}^{dr(b-1)}$ for $j \geq 0$ (see Example 5.20). As an even more specific example, the probability that a polynomial (of degree at least 4) over \mathbb{F}_q has two double roots “or worse” (a quadruple root; a triple root is *not* enough) is q^{-2} .

1.37. As a further example, if v has all distinct elements greater than 1, then

$$\lim_{j \rightarrow \infty} \frac{[w_{1jv}(X)]}{[\mathrm{Sym}^{j+\sum v} X]} = \frac{[w_v(X)]}{\zeta_X(2d)} \frac{\mathbb{L}^{-d \sum v}}{(1 + \mathbb{L}^{-d})^{|v|}}.$$

(See Example 5.13.) Using the “fibration” $\alpha : w_{1jv} \rightarrow w_v$, one can give a “fiberwise heuristic” which yields this as a prediction. But because α is not a fibration in the Zariski topology, this heuristic does not give a proof, so we omit the details.

1.38. Our last specific example is the following. Let $\text{Sym}_s^j X$ (not to be confused with $\text{Sym}_{[s]}^j X$, §1.6) be the locally closed subset of $\text{Sym}^j X$ corresponding to collections of points containing *exactly* s multiple points.

1.39. **Theorem.** — Suppose X is a geometrically irreducible variety of dimension d .

(a) If X satisfies MSSP_ϕ , then

$$\lim_{j \rightarrow \infty} \frac{[\text{Sym}_s^j X]}{\mathbf{L}^{jd}} = \frac{\zeta_X^{[s]}(2d)}{\zeta_X(2d)} \text{SP}_\phi(X) \quad \text{in } \phi(\widehat{\mathcal{M}_L}).$$

If furthermore the $[\text{Sym}^j X]$ are invertible in $\phi(\widehat{\mathcal{M}_L})$ (e.g. if X is rational or $\phi = \text{HS}$, §1.5), then

$$(1.40) \quad \lim_{j \rightarrow \infty} \frac{[\text{Sym}_s^j X]}{[\text{Sym}^j X]} = \frac{\zeta_X^{[s]}(2d)}{\zeta_X(2d)} \quad \text{in } \phi(\widehat{\mathcal{M}_L}).$$

(b) If $\mathbb{K} = \mathbb{F}_q$,

$$\lim_{j \rightarrow \infty} \frac{\#\text{Sym}_s^j X}{\#\text{Sym}^j X} = \frac{\zeta_X^{[s]}(2d)}{\zeta_X(2d)}.$$

The proof of Theorem 1.39 concludes just after the statement of Theorem 5.10.

The similarity of (1.40) to Theorem 1.13 is striking. In fact, for $X = \mathbb{P}^1$, Theorem 1.13 and Theorem 1.39 give the same result, but for a smooth curve C of arbitrary genus, Theorem 1.39 gives the limit of the moduli spaces of *all* divisors, and Theorem 1.13 gives the analogous result for moduli spaces of divisors in multiples of a fixed linear system (although the answer does not depend on the linear system). Thus although in the case of (smooth projective geometrically irreducible) curves of positive genus, (1.40) and Theorem 1.13 are logically independent, they are consistent in some strong sense.

1.41. Connections to configuration spaces in topology.

We now draw connections to topological work.

1.42. The “contractible” case $X = \mathbb{A}^d$. We begin with the case where $X = \mathbb{A}^d$, to highlight the topology arising from the positions of the points rather than the underlying space. Note first that $[\text{Sym}^r \mathbb{A}^d] = [\mathbb{A}^{rd}]$ (a fact first proved by Totaro, [G2, Lemma 4.4], see also [GZLMH3, Thm. 1] and [GZLMH1, Statement 2]), so $\mathbf{Z}_X(t) = 1/(1 - [X]t) = 1/(1 - \mathbf{L}^d t)$.

Proposition 5.9(b) implies that $[\overline{w}_{1^d}(\mathbb{A}^d)] = \mathbf{L}^{d(j+1)}$. Occam’s Razor 1.2 then gives a striking prediction (the case $r = 0$ of Conjecture 1.43 below). The results given in §1.36, and more generally Example 5.20, suggest even more (the full statement of Conjecture 1.43 below).

Before stating it, we point out that for an arbitrary complex manifold X , it is more natural to study the complement $\overline{w}_\lambda^c(X)$ of $\overline{w}_\lambda(X)$ in $\text{Sym}^{\sum_i \lambda_i}(X)$. For example, $\overline{w}_{1^2}^c(X) =$

$w_{1j+2}(X)$. The spaces $\overline{w}_\lambda^c(X)$ satisfy Poincare duality (with \mathbb{Q} -coefficients, see Motivation 1.26(v)).

1.43. Conjecture E. — *If $1 < a \leq b$, and j and r are nonnegative integers, then $h_i(\overline{w}_{1jabr}^c(\mathbb{A}_{\mathbb{C}}^d), \mathbb{Q})$ is 1 if $i = 0$ or $i = 2d((a-1) + r(b-1)) - 1$, and 0 otherwise.*

1.44. In the case when $d = 1$, $a = 2$, $r = 0$ (hence b arbitrary), and j arbitrary, this is a result of Arnol'd [A1]. In the case when $d = 1$, $a = b$, and r and j are arbitrary, this is a consequence of [A2, (19) and (20)] (but note a mistake in the formulation of [A2, (23)]). T. Church explained this to us, and explained how Arnol'd's proofs of these cases extends to general d , [Ch3]. O. Randal-Williams [RW1] has also proved the case of one multiple point (i.e. $r = 0$, and j , a , b , and d arbitrary) using [Kal].

Given Example 5.20, the reader may suspect that $\overline{w}_\lambda(\mathbb{A}^d)$ is always a power of \mathbf{L} for all λ , but this is not the case. The smallest λ for which this is false is $\lambda = 1^2 2^2 3$ (see the last line of §2). In fact, as $j \rightarrow \infty$, $\overline{w}_{1j223}(\mathbb{A}_{\mathbb{C}}^d)$ has an unbounded number of nonzero cohomology groups with compact support; this can be seen through a calculation of the generating series $\sum_j [\overline{w}_{1j223}(X)] t^j$ using Theorem 5.2.

1.45. General X . The stabilization of the Betti numbers (in fact the integral homology) of $w_{1j}(X)$ for open manifolds X was proven by McDuff [Mc]. Recently, Church [Ch2, Cor. 3] and Randal-Williams [RW2] proved the stabilization of the Betti numbers $h_k(w_{1j}(X), \mathbb{Q})$ for closed, connected manifolds X of finite type. This is the topological analog of the motivic limits existing in our first example: $\lim_{j \rightarrow \infty} [w_{1j}(X)] / \mathbf{L}^{jd}$ (cf. Occam's Razor 1.2). Upon hearing of our result for the motivic stabilization of partially labeled configuration spaces $[w_{1j\lambda}(X)] / \mathbf{L}^{-jd}$, Church [Ch2, Thm. 5] and Randal-Williams [RW1] also proved the stabilization of the Betti numbers of these spaces for manifolds X .

We conjecture stabilization of Betti numbers for the other flavors of configuration spaces that have motivic limits (Theorems 1.30 and 1.39).

1.46. Conjecture F. — *Given i and a partition λ , for an irreducible smooth complex variety X , the limit $\lim_{j \rightarrow \infty} h_i(\overline{w}_{1j\lambda}^c(X), \mathbb{Q})$ exists.*

See §1.44 for the case $X = \mathbb{A}_{\mathbb{C}}^d$ and $\lambda = m^r$ (m and r arbitrary). (As in the case of $w_{1j}(X)$, there do not exist obvious maps among the elements of these sequences of configuration spaces for closed X ; many topological stabilization results rely on such a map.) One might ask a similar question for the constructible subset $\text{Sym}_s^j X \subset \text{Sym}^j X$, for each s (cf. Theorem 1.39).

The formulas of Theorem 1.30 (given recursively in Propositions 5.11 and 5.14(b)) can be combined with the formulas for the Hodge-Deligne series of zeta functions [C2, Prop. 1.1] to obtain explicit formulas for the Hodge-Deligne series of the limits of (various flavors of) configuration spaces above $(w_{1j\lambda}(X), \overline{w}_{1j\lambda}^c(X), \text{Sym}_s^j X)$ in terms of the Hodge-Deligne

polynomial of X . Totaro [T] gives an explicit spectral sequence with only one non-trivial differential that computes the Betti numbers of the usual configuration spaces, but this does not immediately give the limit Poincaré series. One should hope to compare the limit Hodge-Deligne series of various configuration spaces to the analogous (mostly unknown) limit Poincaré series. In particular, in the situations in this paper, we not only know that limit Hodge-Deligne series exist, but we have given relatively simple formulas for them. Are there analogous simple formulas for the limit Poincaré series?

More precisely, in analogy with Conjectures C and D, for all the flavors of configuration spaces we discuss, we conjecture the limiting Poincaré series is finite when the analogous limiting motive has finite Hodge-Deligne series, and wonder whether, in these cases, Occam's Razor 1.2 predicts the correct Betti numbers. As stated in §1.44, in the case $X = \mathbb{A}_{\mathbb{C}}^d$, for \overline{w}_{1j2} , Arnol'd has shown the answer is yes; and for $X = \mathbb{P}_{\mathbb{C}}^1$, for w_{1j} , Church has done the same, [Ch1, Prop. 4.5].

As an open example, applying Occam's Razor 1.2 to Example 1.34 in the case $X = \mathbb{P}_{\mathbb{C}}^2$ yields the following prediction.

1.47. Conjecture G. — *We have*

$$\lim_{j \rightarrow \infty} h_i(w_{1j}(\mathbb{P}_{\mathbb{C}}^2), \mathbb{Q}) = \begin{cases} 1 & \text{if } i = 0, 2, 4, 7, 9, 11 \\ 0 & \text{otherwise.} \end{cases}$$

1.48. We conjecture that the limiting Poincaré series is periodic when the analogous limiting motive has periodic Hodge-Deligne series, as in the following example, which is about the space of configurations with precisely one double point.

1.49. Conjecture H. — *The limits $\lim_{j \rightarrow \infty} h_i(w_{1j2}(\mathbb{A}_{\mathbb{C}}^d), \mathbb{Q})$ are periodic in i .*

If we further apply Occam's Razor 1.2 to our results, it would predict that for each i ,

$$(1.50) \quad \lim_{j \rightarrow \infty} h_i(w_{1j2}(\mathbb{A}_{\mathbb{C}}^d), \mathbb{Q}) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 2(2k-1)d - 1 \text{ or } 4kd, \text{ for } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

O. Randal-Williams has suggested that Conjecture H is true but that the prediction (1.50) is false, [RW1].

1.51. Connections to configuration spaces in number theory.

The limits in Theorems 1.30 and 1.39 have natural analogs over \mathbb{Z} . For a partition $\nu = [e_1, e_2, \dots, e_k]$, we say an integer n has *at least ν -power* if $\prod_i a_i^{e_i} | n$ for some (not necessarily distinct) integers $a_i > 1$. The limit resulting from the generating function in Example 5.20

then has the following analog over \mathbb{Z} :

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N \mid n \text{ has at least } ab^r\text{-power}\}}{N} \\ = 1 - \frac{1}{\zeta(-b)} \left(\sum_{i=0}^{r-1} \sum_{p_1 \leq \dots \leq p_i} p_1^{-b} \cdots p_i^{-b} \right) - \sum_{p_1 \leq \dots \leq p_r} p_1^{-b} \cdots p_r^{-b} \frac{1}{\zeta(-a)},$$

where the sums above are over primes p_j . Theorems 1.30 and 1.39 also suggest natural point counting analogs for arithmetic schemes (as in [P2, §5]). One expects that when X is a general arithmetic scheme, such results, as in [P2, §5], will require new ideas.

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2. NOTATION FOR PARTITIONS AND CONFIGURATION SPACES

Our arguments use partitions in a slightly more general sense than usual. For us, a (*generalized*) *partition* in an abelian semigroup S is a finite multiset of elements of S . A *subpartition* is a submultiset of a partition. Partitions in the traditional sense are the special case $S = \mathbb{Z}_{>0}^+$. We use the standard notation $\mu \vdash n$ (“ μ is a partition of n ”). Let \mathcal{P} be the set of all partitions in the traditional sense (i.e. of positive integers). We use the notation $[\dots]$ to denote a multiset, exponents to denote multiplicity, and concatenation to denote union, so for example $a^2b = a[a, b] = [a][a, b] = [a, a, b]$.

In §3, we will want to concatenate λ and μ and consider the parts of λ and μ to be distinct (“disjoint concatenation”), which may require some renaming; we write this as $\lambda \cdot \mu$. For example, we may write $[a, b] \cdot [a, a]$ as $[a_1, b_1, a_2, a_2]$.

As usual, $|\lambda|$ denotes the number of elements of the multiset λ . A generalized partition λ has a *multiplicity partition* $m(\lambda)$ of $|\lambda|$ — for example $m([a, a, b]) = [2, 1]$. We write $\|\lambda\|$ for the number of *distinct* elements of λ (so for example $\|\mu\| = m$ for $\mu \in \mathcal{Q}(m)$). We write $\sum \lambda$ for the *sum* of the generalized partition λ , i.e. $\sum_{s \in \lambda} s$. Clearly, for any generalized partition, $|\lambda| = \sum m(\lambda)$ and $\|\lambda\| = |m(\lambda)|$. For example, suppose $\lambda = 1^3 2^3 3 4^2 5$, so $m(\lambda) = [3, 3, 1, 2, 1]$. Then $\sum \lambda = 25$, $|\lambda| = \sum m(\lambda) = 10$, $\|\lambda\| = |m(\lambda)| = 5$, and $\|m(\lambda)\| = 3$.

Let $\mathcal{Q}(m)$ be the set of partitions in the traditional sense in which exactly the numbers 1 through m appear. (We also think of $\mathcal{Q}(m)$ as partitions of m linearly ordered

elements, up to isomorphisms of the ordered elements.) By taking the multiplicity partitions, we can interpret $\mathcal{Q}(m)$ as the *ordered* partitions with exactly m parts. For example, $[1, 1, 1, 1, 2, 3, 3] = 1^4 2^1 3^2 \in \mathcal{Q}(3)$ can be reinterpreted as $4 + 1 + 2 = 7$. Let $\mathcal{Q} = \cup_m \mathcal{Q}(m)$, which can be reinterpreted as the set of all ordered partitions.

Suppose λ and λ' are generalized partitions in S . If there are sub(multi)sets $[x, y] \subset \lambda$ and $[z] \subset \lambda'$ such that $x + y = z$ and $\lambda \setminus [x, y] = \lambda' \setminus [z]$, we say λ' is an *elementary merge* of λ . In this case $|\lambda| = 1 + |\lambda'|$. We define the *refinement ordering* $<$ on generalized partitions in S as generated by elementary merges. (If λ' is an elementary merge of λ , then $\lambda < \lambda'$.) For example, $[1, 2, 3] < [3, 3] < [6]$. We write $\lambda \leq \lambda'$ if $\lambda < \lambda'$ or $\lambda = \lambda'$.

Given a generalized partition $\lambda = [\lambda_i]$, define the *formalization* as $f(\lambda) := [\alpha_{\lambda_i}]$ (in the abelian semigroup $\mathbb{Z}^+[\alpha_i]_{i \in S}$); we have replaced entries with “formal” replacements. The purpose of this construction is to obtain a partition with the same multiplicity sequence such that for all $S_1, S_2 \subset \lambda$ such that $\sum S_1 = \sum S_2$, we have $S_1 = S_2$.

If λ is a generalized partition, define $\text{Sym}^\lambda X = \prod_{m_i \in m(\lambda)} \text{Sym}^{m_i} X$. For example, $\text{Sym}^{[a, a, b]} X$ parametrizes an unordered pair of (geometric, not necessarily distinct) points of X labeled a , and another point (not necessarily distinct) labeled b . (Warning: do not confuse $\text{Sym}^{[2]} X$ with $\text{Sym}^2 X$: by definition $\text{Sym}^{[2]} X = X$.) We define $w_\lambda(X)$ (or simply w_λ for convenience) to be the open subscheme of $\text{Sym}^\lambda X$ in which all the points are distinct, i.e. the complement of the “big diagonal”. (This generalizes the definition of w_λ given at the start of §1.29, which is the case of traditional partitions.) For example, $w_{[a, a, b]}(X)$ parametrizes an unordered pair of distinct points of X labeled a , along with a third distinct point, labeled b . Note that w_λ depends only on the multiplicity sequence $m(\lambda)$.

Define $\overline{w}_\lambda = \sum_{\lambda \leq \mu} [w_\mu]$. Although \overline{w}_λ is defined as an element of \mathcal{M} , we can often naturally endow it with the structure of a variety, as the closure of w_λ in an appropriate space. For example, if λ is a traditional partition ($S = \mathbb{Z}^+$), then \overline{w}_λ is the class of the closure of w_λ in $\text{Sym}^{\sum \lambda} X$; thus this definition of \overline{w}_λ generalizes the one given at the start of §1.29. The varieties w_λ and \overline{w}_λ have been studied by Haiman and Woo (see Z_λ° and Z_λ in [HW, §3.2]). If λ is a formalization, since w_λ is the open subset of $\text{Sym}^\lambda X$ where the $|\lambda|$ points are distinct, and the various μ with $\lambda < \mu$ correspond to letting the points come together in various ways, we have

$$(2.1) \quad \overline{w}_\lambda(X) = [\text{Sym}^\lambda X] \quad \text{for a formalization } \lambda.$$

But (2.1) need not hold if λ is not a formalization. As perhaps the simplest example, if $\lambda = [1, 1, 2, 2, 3]$, then $\overline{w}_\lambda(\mathbf{L}) = \mathbf{L}^5 - \mathbf{L}^2 + \mathbf{L}$, while $\text{Sym}^\lambda \mathbf{L} = \mathbf{L}^5$ (using $\text{Sym}^n \mathbf{L} = \mathbf{L}^n$).

3. MODULI OF HYPERSURFACES

The goal of this section is to prove Theorem 1.13. Throughout this section, X is assumed to be smooth of pure dimension d . In order to prove Theorem 1.13 in general, we first establish it for $s = 0$. This case will be completed by Proposition 3.8, see §3.7. We determine the motive of smooth divisors in a linear system by considering all divisors, and removing those with singularities.

Suppose λ is a generalized partition and \mathcal{F} is a line bundle on X . We define three types of incidence subschemes parametrizing sections of \mathcal{F} singular at points marked by λ .

In analogy with the notation w_λ , let $W_\lambda(\mathcal{F})$ (or W_λ when \mathcal{F} is clear from context) denote the locally closed subvariety of $H^0(X, \mathcal{F}) \times w_\lambda(X)$ corresponding to sections of \mathcal{F} singular at *precisely* those $|\lambda|$ (necessarily distinct) geometric points of X given by the point of $w_\lambda(X) \subset \text{Sym}^\lambda X$. For example, $W_{*^s}(\mathcal{F}) \cong H^0(X, \mathcal{F})^s$.

Let $W_{\geq \lambda} = W_{\geq \lambda}(\mathcal{F})$ be the locally closed subvariety of $H^0(X, \mathcal{F}) \times w_\lambda(X)$ corresponding to sections of \mathcal{F} singular at those $|\lambda|$ (necessarily distinct) geometric points of X given by the point of $w_\lambda(X) \subset \text{Sym}^\lambda X$, *and possibly elsewhere*. Note that W_λ is an open subset of $W_{\geq \lambda}$.

If k is a nonnegative integer, let $W_{\lambda, \geq k} = W_{\lambda, \geq k}(\mathcal{F})$ be the locally closed subset of $H^0(X, \mathcal{F}) \times w_\lambda(X)$ corresponding to those (s, t) for which s is singular at the $|\lambda|$ points parametrized by t and *at least k additional geometric points*. Because $W_{\lambda, \geq k}$ is the image of $W_{\geq \lambda, *^k}$ (disjoint concatenation “ \cdot ” was defined in §2) under the obvious projection, $W_{\lambda, \geq k}$ is a constructible subset of $H^0(X, \mathcal{F}) \times w_\lambda(X)$ by Chevalley’s theorem, and thus has a well-defined class in \mathcal{M} . (This also follows from (3.1) below.)

Clearly

$$(3.1) \quad [W_{\geq \lambda}] = [W_\lambda] + [W_{\lambda, \geq 1}] = [W_\lambda] + [W_{\lambda, *}] + [W_{\lambda, \geq 2}] = [W_\lambda] + [W_{\lambda, *}] + [W_{\lambda, **}] + [W_{\lambda, \geq 3}] = \cdots$$

For example, a section singular at some points labeled by λ is: (0) nonsingular elsewhere, or else (i) singular at precisely one point elsewhere, or else (ii) singular at precisely two points elsewhere, or else (iii) singular at 3 or more other points elsewhere.

3.2. Lemma. — *With \mathcal{L} ample and fixed, and j sufficiently large in terms of $|\lambda|$, we have that $W_{\geq \lambda}(\mathcal{L}^{\otimes j})$ is a vector bundle over $w_\lambda(X)$ of rank $r - |\lambda|(d + 1)$, where $r = h^0(X, \mathcal{L}^{\otimes j})$.*

Proof. The following argument will not surprise experts, but we include it for completeness. The result is insensitive to base field extension, so we assume $\mathbb{K} = \overline{\mathbb{K}}$. The scheme $W_{\geq \lambda}$ corresponds to a coherent sheaf on $w_\lambda(X)$, corresponding to sections of $\mathcal{L}^{\otimes j}$ singular at the $|\lambda|$ points parametrized by $\text{Sym}^\lambda X$. We wish to show that this coherent sheaf is a vector bundle of rank $h^0(X, \mathcal{L}^{\otimes j}) - |\lambda|(d + 1)$.

By Grauert's Theorem, it suffices to show that (for $j \gg_{|\lambda|} 0$) for any closed point of $w_\lambda(X)$, interpreted as $|\lambda|$ distinct points of X , the 1-jets at the points impose independent conditions on sections of $\mathcal{L}^{\otimes j}$. Now choose j so $\mathcal{L}^{\otimes j}$ is $(3|\lambda|)$ -very ample. \square

The typical Noetherian induction using the local triviality of vector bundles yields, for $j \gg_{|\lambda|} 0$,

$$(3.3) \quad [W_{\geq \lambda}] = [w_\lambda] \mathbb{L}^{r-|\lambda|(d+1)}.$$

3.4. Corollary (and definition of j_N). — *Fix an ample line bundle \mathcal{L} on X of dimension $d > 0$. For each positive integer N , there is some j_N , so that for $j \geq j_N$ (where as in Lemma 3.2, $r = h^0(X, \mathcal{L}^{\otimes j})$):*

- (a) $r > 2N$;
- (b1) $W_{\geq \lambda}(\mathcal{L}^{\otimes j})$ (and hence its open subset $W_\lambda(\mathcal{L}^{\otimes j})$) has (pure) dimension $\dim r - |\lambda|$ for $|\lambda| \leq N$;
- (b2) $W_{\geq \lambda}(\mathcal{L}^{\otimes j})$ (and hence its open subset $W_\lambda(\mathcal{L}^{\otimes j})$) has dimension less than $r - N$ for $|\lambda| \geq N + 1$;
- (c1) $W_{\lambda, \geq k}$ has dimension at most $r - |\lambda| - k$ for $|\lambda| + k \leq N$;
- (c2) $W_{\lambda, \geq k}$ has dimension less than $r - N$ for $|\lambda| + k \geq N + 1$; and
- (d) j is sufficiently large (in the sense of Lemma 3.2) for all partitions of integers of length at most $N + 1$.

Proof. Let j_N be sufficiently large (in the sense of Lemma 3.2) for all partitions of length at most $N + 1$ which gives (d). Part (b1) is clear from Lemma 3.2. For part (b2), choose a subpartition $\lambda' \subset \lambda$ of length $N + 1$. Then the result follows from $W_{\geq \lambda}(\mathcal{L}^{\otimes j}) \subset W_{\geq \lambda'}(\mathcal{L}^{\otimes j})$. Parts (c1) and (c2) follows from the surjectivity of $W_{\geq \lambda, *k} \rightarrow W_{\lambda, \geq k}$, and (b1) and (b2). Taking j_N even larger, (a) can clearly be satisfied, by ampleness of \mathcal{L} . \square

We will prove Theorem 1.13 by showing it modulo “codimension $N + 1$ ” (i.e. modulo those classes of dimension at most $-(N + 1)$ in \mathcal{M}_L) for each N , for $j \geq j_N$. In what follows, $\mathcal{F} = \mathcal{L}^{\otimes j}$ where $j \geq j_N$.

Modulo dimension $< r - N$,

$$\begin{aligned} W_\lambda &\equiv W_{\geq \lambda} - \sum_{k_1=1}^{\infty} W_{\lambda, *k_1} \\ &\equiv W_{\geq \lambda} - \sum_{k_1=1}^{\infty} W_{\geq \lambda, *k_1} + \sum_{k_1, k_2=1}^{\infty} W_{\lambda, *k_1 \bullet k_2} \\ &\equiv W_{\geq \lambda} - \sum_{k_1=1}^{\infty} W_{\geq \lambda, *k_1} + \sum_{k_1, k_2=1}^{\infty} W_{\geq \lambda, *k_1 \bullet k_2} - \sum_{k_1, k_2, k_3=1}^{\infty} W_{\lambda, *k_1 \bullet k_2 \bullet k_3} \\ &\equiv \dots \end{aligned}$$

Continuing (i.e. by an easy induction), we may write W_λ in terms of $W_{\geq \mu}$ for various μ as follows. Then

$$(3.5) \quad W_\lambda \equiv \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} W_{\geq \lambda \mu} \equiv \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} w_{\lambda \mu} \mathbf{L}^{r-|\lambda \mu|(d+1)} \pmod{\dim < r - N} \quad (\text{by (3.3)})$$

in \mathcal{M} . We have proved the following.

3.6. Proposition. — *For any generalized partition λ , integer $N \geq |\lambda|$, and $j \geq j_N$,*

$$\frac{W_\lambda}{\mathbf{L}^r} \equiv \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} \frac{w_{\lambda \mu}}{\mathbf{L}^{|\lambda \mu|(d+1)}} \pmod{\text{codim} > N}.$$

Hence modulo codimension $> N$, for $j \geq j_N$, the motivic probability of sections of $\mathcal{L}^{\otimes j}$ being smooth is

$$\frac{W_\emptyset}{\mathbf{L}^r} = \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} \frac{w_\mu}{\mathbf{L}^{|\mu|(d+1)}} \in \widehat{\mathcal{M}}_{\mathbf{L}}.$$

We have thus shown that for $j \geq j_N$, the left side of (1.14) (in the case $s = 0$) stabilizes up to codimension N , to the expression of Proposition 3.6 for $\lambda = \emptyset$. We compare this to the right side of (1.14). We have

$$\frac{1}{Z_X(t)} = \frac{1}{\sum_{k=0}^{\infty} [\text{Sym}^k X] t^k} = \sum_{m=0}^{\infty} \left(- \sum_{k=1}^{\infty} (\text{Sym}^k X) t^k \right)^m = \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} [\text{Sym}^\mu X] t^{|\mu|}.$$

3.7. The $s = 0$ case of Theorem 1.13 is then a consequence of the following proposition.

3.8. Proposition. — *We have*

$$\sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} w_\mu t^{|\mu|} = \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} [\text{Sym}^\mu X] t^{|\mu|},$$

and hence

$$(3.9) \quad \frac{1}{Z_X(t)} = \sum_{\mu \in \mathcal{Q}} (-1)^{|\mu|} w_\mu t^{|\mu|}.$$

Notice that $\text{Sym}^\mu X$ includes w_μ plus smaller-dimensional contributions (cf. (2.1)). Thus Proposition 3.8 states that “the smaller terms cancel”.

Proof. For $\mu \in \mathcal{Q}$, we can expand

$$\text{Sym}^\mu X = \text{Sym}^{f(\mu)} X = \overline{w}_{f(\mu)} = \sum_{\lambda \geq f(\mu)} w_\lambda.$$

Thus we have

$$\begin{aligned} \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} [\text{Sym}^\mu X] t^{|\mu|} &= \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} \sum_{\lambda \geq f(\mu)} w_\lambda t^{|\mu|} \\ &= \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} w_\mu t^{|\mu|} + \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} \sum_{\lambda > f(\mu)} w_\lambda t^{|\mu|}. \end{aligned}$$

We prove the second term in the previous line is 0 by finding a bijection, for a fixed $m(\lambda)$ (hence fixed w_λ) and fixed $|\mu|$, between terms with odd $\|\mu\|$ and terms with even $\|\mu\|$. The map is as follows. We map a pair (λ, μ) with $\lambda > f(\mu)$ to a pair (λ', μ') with $\lambda' > f(\mu')$ that will be constructed below.

The elements of λ are sums of elements from $f(\mu)$, i.e. of $a_1, \dots, a_{\|\mu\|}$. Write each element of λ as $a_{i_1} + \dots + a_{i_s}$ with $i_1 \leq \dots \leq i_s$. We say such an element has length s . Among the longest elements of λ , take the lexicographically first sum, call them the *top sums* (there may be a tie) and suppose it ends with a_k (i.e. includes at least one a_k term and no a_j term for $j > k$). If each of the top sums has exactly one a_k , and there are no other a_k 's in any other terms of λ (which, together, in particular implies that $k \geq 2$), then we are in **case 1**. Otherwise, (if a top sum has at least 2 a_k terms, or there are non-top-sum elements of λ containing an a_k summand) we are in **case 2**.

If we are in case 1, note that $k \geq 2$. As a first attempt, we construct λ' by turning all of the a_k 's in λ into a_{k-1} 's, and we construct μ' by changing all the k 's in μ into $(k-1)$'s. However, with this construction μ' would not necessarily be a partition composed of consecutive integers starting with 1. So in fact, for each $j \geq k$, we turn all of the j 's appearing in μ and λ to $(j-1)$'s (as elements or subscripts) to obtain μ' and λ' , respectively. We have $\lambda' > f(\mu')$ and $\|\mu'\| = \|\mu\| - 1$. Since the top sum was lexicographically first among the longest sums, when we replaced a_k by a_{k-1} we do not make any elements of λ equal that were not equal before, and thus $m(\lambda) = m(\lambda')$. Clearly, $|\mu'| = |\mu|$. Furthermore, note that the top sums of λ were changed into elements of λ' which are now top sums of λ' . However, since there had to be an a_{k-1} appearing somewhere in the original λ , the new (λ', μ') we have created is in case 2.

In case 2, for all $j > k$ we change all the j 's in λ and μ (including in subscripts) to $j+1$'s. Then we also change one a_k in each top sum to an a_{k+1} to obtain λ' , and we change the same number of k 's from μ into $(k+1)$'s to obtain μ' . We have $\lambda' > f(\mu')$ and $\|\mu'\| = \|\mu\| + 1$. Since we change all of the top sums in the same way, we don't make any terms of λ unequal that were previously equal, and thus $m(\lambda) = m(\lambda')$. Clearly, $|\mu'| = |\mu|$. Furthermore, note that the top sums were changed into elements of λ which are now top sums of the new λ' . However, since the a_{k+1} 's appear in λ' only in the top sums, and only once in each top sum, the new chain we have created is in case 1. If we then applied the map again, we can see we will get back to (λ, μ) . Similarly, we can check that if we

apply the map twice to a (λ, μ) in case 1, we also get back to the original chain. Thus this operation is an involution, and thus a bijection between even $\|\mu\|$ and odd $\|\mu\|$ terms. \square

3.10. Proof of Theorem 1.13 in general (singularities). In analogy with the inverse of the motivic zeta function $Z_X^{-1}(t)$, and in light of Proposition 3.8, we define

$$(3.11) \quad Z_{X,\lambda}^{-1}(t) := \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} w_{\lambda \cdot \mu} t^{|\lambda \cdot \mu|} \in \mathcal{M}[[t]],$$

(a function of X). For example, $Z_{X,\emptyset}^{-1}(t) = Z_X^{-1}(t)$ by (3.9).

We will deduce Theorem 1.13 from Proposition 3.12 and Lemma 3.18, both of which will require some time to prove. We combine them to prove Theorem 1.13 in §3.23.

3.12. Proposition. — *There exist $c_{\pi,\lambda} \in \mathbb{Z}[[t]]$ (independent of X and \mathbb{K}), such that*

$$(3.13) \quad Z_{X,\lambda}^{-1}(t) = \sum_{\substack{\pi \in \mathcal{P} \\ |\pi| = |\lambda|}} c_{\pi,\lambda} w_{\pi} Z_X^{-1}(t).$$

Proof. Suppose $\lambda = \prod a_i^{r_i}$, and μ is a partition whose parts are distinct from the a_i . Then we have a *product rule* (3.14) for $w_{\lambda \cdot \mu}$ in terms of $w_{\lambda} w_{\mu}$ and “lower order terms”. Clearly $w_{\lambda} w_{\mu}$ (interpreted as configurations of distinct points labeled by λ and distinct points labeled by μ) can be interpreted as the union (or sum) of $w_{\lambda \cdot \mu}$ along with loci where some of the points of λ overlap with some of the points of μ . We thus have the following formula, where $\mu(i)$ is a subpartition of μ corresponding to which points of μ overlap with the a_i -labeled points of λ .

$$(3.14) \quad [w_{\lambda \cdot \mu}] = [w_{\lambda}] [w_{\mu}] - \sum_{\substack{\mu(i) \text{ not all empty} \\ \cup_i \mu(i) \subset \mu \\ |\mu(i)| \leq r_i}} \left[w_{(a_1^{r_1 - |\mu(1)|} a_2^{r_2 - |\mu(2)|} \dots) \cdot (\mu(1)\mu(2)\dots) \cdot (\mu \setminus \cup \mu(i))} \right].$$

Now we sum the product rule over all $\mu \in \mathcal{Q}$ to obtain

$$\begin{aligned} Z_{X,\lambda}^{-1}(t) &= \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} [w_{\lambda \cdot \mu} t^{|\lambda \cdot \mu|}] \\ &= \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} [w_{\lambda}] [w_{\mu} t^{|\mu|}] \\ &\quad - \sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \text{ not all empty} \\ \cup_i \mu(i) \subset \mu \\ |\mu(i)| \leq r_i}} (-1)^{\|\mu\|} \left[w_{(a_1^{r_1 - |\mu(1)|} a_2^{r_2 - |\mu(2)|} \dots) \cdot (\mu(1)\mu(2)\dots) \cdot (\mu \setminus \cup \mu(i))} t^{|\lambda \cdot \mu|} \right] \\ &= [w_{\lambda}] t^{|\lambda|} Z_X^{-1}(t) \\ &\quad - \sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \text{ not all empty} \\ \cup_i \mu(i) \subset \mu \\ |\mu(i)| \leq r_i}} (-1)^{\|\mu\|} \left[w_{(a_1^{r_1 - |\mu(1)|} a_2^{r_2 - |\mu(2)|} \dots) \cdot (\mu(1)\mu(2)\dots) \cdot (\mu \setminus \cup \mu(i))} \right] t^{|\lambda \cdot \mu|} \quad (\text{by (3.9)}). \end{aligned}$$

In this sum, we will group together all the terms where the $\mu(i)$ have some fixed multiplicity sequence $\sigma(i)$, which is an ordered partition. Let $F(\sigma(i))$ be a fixed partition with multiplicity sequence $\sigma(i)$. For typographical simplicity, let $B = \left(a_1^{r_1 - \sum \sigma(1)} a_2^{r_2 - \sum \sigma(2)} \dots \right) \cdot (F(\sigma(1))F(\sigma(2)) \dots)$.

$$\begin{aligned}
 \mathbf{Z}_{X,\lambda}^{-1}(t) &= [w_\lambda] t^{|\lambda|} \mathbf{Z}_X^{-1}(t) \\
 &\quad - \sum_{\substack{\sigma(i) \\ 0 \leq' \sum \sigma(i) \leq r_i}} \sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \\ \cup \mu(i) \subset \mu \\ m(\mu(i)) = \sigma(i)}} (-1)^{\|\mu\|} \left[w_{(a_1^{r_1 - |\mu(1)|} a_2^{r_2 - |\mu(2)|} \dots) \cdot (\mu(1)\mu(2)\dots) \cdot (\mu \setminus \cup \mu(i))} \right] t^{|\lambda\mu|} \\
 &= [w_\lambda] t^{|\lambda|} \mathbf{Z}_X^{-1}(t) \\
 (3.15) \quad &\quad - \sum_{\substack{\sigma(i) \\ 0 \leq' \sum \sigma(i) \leq r_i}} t^{|\sum_i (\sum \sigma(i))|} \sum_{\pi \in \mathcal{Q}} (-1)^{\|\pi\|} [w_{B \cdot \pi}] t^{|B\pi|} \sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \\ \cup \mu(i) \subset \mu \\ m(\mu(i)) = \sigma(i) \\ \mu \setminus \cup \mu(i) \sim \pi}} (-1)^{\|\mu\| - \|\pi\|},
 \end{aligned}$$

where $0 \leq' \sigma(i)$ means that not all $\sum \sigma(i)$ may be 0, and \sim stands for isomorphism of partitions with linearly ordered elements.

We now apply the following lemma, whose proof we defer for a few paragraphs.

3.16. Lemma. — *Given ordered partitions $\sigma(i) = [\sigma(i)_j]_j$, with multiplicity sequence $a(i) = [a(i)_j]_j$, and a partition π with linearly ordered elements, we have*

$$\sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \\ \cup \mu(i) \subset \mu \\ m(\mu(i)) = \sigma(i) \\ \mu \setminus \cup \mu(i) \sim \pi}} (-1)^{\|\mu\| - \|\pi\|} = \prod_i (-1)^{|\sigma(i)|} \frac{|\sigma(i)|!}{\prod_j a(i)_j!}.$$

Thus we have

$$(3.17) \quad \mathbf{Z}_{X,\lambda}^{-1}(t) = [w_\lambda] t^{|\lambda|} \mathbf{Z}_X^{-1}(t) - \sum_{\substack{\sigma(i) \\ 0 \leq' \sum \sigma(i) \leq r_i}} \left(\prod_i (-1)^{|\sigma(i)|} \frac{|\sigma(i)|!}{\prod_j a(i)_j!} \right) t^{|\sum_i \sum \sigma(i)|} \mathbf{Z}_{X,B}^{-1}(t).$$

Note that $m(B) \leq m(\lambda)$ in the merge ordering. We assume that we know $\mathbf{Z}_{X,B}^{-1}$ inductively for $m(B) < m(\lambda)$. If $m(B) = m(\lambda)$, then $\mathbf{Z}_{X,B}^{-1} = \mathbf{Z}_{X,\lambda}^{-1}$, and we may collect those terms of the left (note that they all have a nonzero power of t appearing with them), and solve for $\mathbf{Z}_{X,\lambda}^{-1}$. In particular, we can prove Proposition 3.12 by inducting on the level of refinement.

To conclude the proof of Proposition 3.12, we prove Lemma 3.16.

Proof of Lemma 3.16. Let ℓ be the number of partitions $\sigma(i)$. We see that

$$\begin{aligned} & \sum_{\mu \in \mathcal{Q}} \sum_{\substack{\mu(i) \\ \cup \mu(i) \subset \mu \\ m(\mu(i)) = \sigma(i) \\ \mu \setminus \cup \mu(i) \sim \pi}} (-1)^{\|\mu\| - \|\pi\|} \\ &= \sum_{\mu_1, \mu_2, \dots, \mu_{\ell-1} \in \mathcal{Q}} \sum_{\substack{\mu(\ell) \subset \mu_{\ell-1} \\ m(\mu(\ell)) = \sigma(\ell) \\ \mu_{\ell-1} \setminus \mu(\ell) \sim \pi}} (-1)^{\|\mu_{\ell-1}\| - \|\pi\|} \dots \sum_{\substack{\mu(2) \subset \mu_1 \\ m(\mu(2)) = \sigma(2) \\ \mu_1 \setminus \mu(2) \sim \mu_2}} (-1)^{\|\mu_1\| - \|\mu_2\|} \sum_{\substack{\mu(1) \subset \mu \\ m(\mu(1)) = \sigma(1) \\ \mu \setminus \mu(1) \sim \mu_1}} (-1)^{\|\mu\| - \|\mu_1\|} \end{aligned}$$

and so we can reduce to the case in which $\ell = 1$, i.e. there is only one partition $\sigma(i)$, which we call σ , with multiplicity sequence $[a_j]_j$.

Fix an integer k and consider the case when $\|\mu\| - \|\pi\| = k$. We will start with π and need to count how many μ with $m(\mu) = \sigma$ we can add to π to obtain a partition with $\|\pi\| + k$ linearly ordered distinct elements. To choose where the new k elements go in the ordering, there are $\binom{\|\pi\| + k}{k}$ possible choices. Then there are $\binom{\|\pi\|}{|\sigma| - k}$ choices for which elements of π will also be elements of μ . Once we have made those choices, there are $\frac{|\sigma|!}{\prod_j a_j!}$ choices for how to assign the $|\sigma|$ multiplicities in σ to these $|\sigma|$ locations. The well-known identity

$$\sum_{k=0}^{|\sigma|} (-1)^k \binom{\|\pi\| + k}{k} \binom{\|\pi\|}{|\sigma| - k} = (-1)^{|\sigma|}.$$

(which can be proved with generating functions, for example), concludes the proof of Lemma 3.16. \square

This in turn concludes the proof of Proposition 3.12. \square

3.18. Lemma (“no unexpected universal linear relations among the w_λ ”). — *The relations $w_\lambda = w_{\lambda'}$ for $m(\lambda) = m(\lambda')$ generate all the \mathbb{Z} -linear relations among the w_λ that hold for all smooth projective varieties X of pure dimension $d > 0$ over all fields \mathbb{K} .*

We conclude the proof of Lemma 3.18 in §3.22.

3.19. Lemma (“no unexpected universal relations among symmetric powers”). — *Suppose we have $f \in \mathbb{Z}[x_1, x_2, \dots]$ such that $f([\text{Sym}^1 X], [\text{Sym}^2 X], \dots) = 0 \in K_0(\text{Var})$ for all smooth projective varieties X of pure dimension over all fields \mathbb{K} . Then $f \equiv 0$.*

In fact, the argument uses only dimension 1.

Proof (Poonen, cf. [P2, §3.3]). Suppose we have a non-zero $f \in \mathbb{Z}[x_1, \dots, x_n]$ (where x_n appears in f) such that $f(\text{Sym}^1 X, \dots, \text{Sym}^n X) = 0$ for all varieties X over any field \mathbb{K} .

Suppose $\mathbb{K} = \mathbb{F}_q$. The information of $\#(\text{Sym}^1 X), \dots, \#(\text{Sym}^n X)$ is equivalent to the information of the number of points of degree 1, \dots , n on X . Thus there exist non-negative integers a_1, a_2, \dots, a_n such that there does not exist an X with a_i points of degree i .

We construct a smooth projective curve with a_i points of degree i for $i = 1, \dots, n$, yielding a contradiction. Choose N large enough so that $\mathbb{P}_{\mathbb{F}_q}^N$ has at least a_i points of degree i , and pick a_i points of degree i in $\mathbb{P}_{\mathbb{F}_q}^N$. Then a standard argument shows that we can find $N - 1$ hypersurfaces f_1, \dots, f_{N-1} intersecting completely (i.e. intersecting in a curve), such that of the points of degree at most n , the f_i pass precisely through our chosen points; and such that the f_i are linearly independent in the tangent space at each of our chosen points. The complete intersection of the f_i is a curve containing precisely the desired number of points of degree at most n , and smooth at those points. Take the normalization of this curve (which will not introduce any more points of small degree). \square

3.20. Observation. We have the following universal formula for $w_\lambda(X)$ in terms of the symmetric powers of X .

$$(3.21) \quad [w_\lambda(X)] = \sum_k \sum_{\lambda = \mu_0 \ll \mu_1 \ll \dots \ll \mu_k} (-1)^k [\text{Sym}^{m(\mu_k)} X].$$

(By Lemma 3.19, this formula is unique.) Here (for the purpose of this argument only) we say $\lambda \ll \lambda'$ if $f(\lambda) < \lambda'$. We show (3.21) by induction on the length of λ . Clearly it is true for $|\lambda| = 1$. Then,

$$\begin{aligned} [w_\lambda(X)] &= [w_{f(\lambda)}(X)] \\ &= [\overline{w}_{f(\lambda)}(X)] - \sum_{f(\lambda) < \mu} [w_\mu(X)] \\ &= [\text{Sym}^{m(f(\lambda))} X] - \sum_{f(\lambda) < \mu} \sum_{\mu = \mu_0 \ll \mu_1 \ll \dots \ll \mu_k} (-1)^k [\text{Sym}^{m(\mu_k)} X]. \end{aligned}$$

3.22. Proof of Lemma 3.18. If we had a finite non-trivial relation $\sum c_\lambda w_\lambda = 0$ only involving terms with distinct multiplicity sequences, then we can use (3.21) to write it as an algebraic relation on $[\text{Sym}^i X]$, holding for all smooth projective X of pure dimension over all fields \mathbb{K} . The terms with non-zero c_λ for maximal $|\lambda|$, will give terms $c_\lambda [\text{Sym}^\lambda X]$, which will be the maximal degree terms in the algebraic relation (where $[\text{Sym}^i X]$ has degree i). Thus we will obtain a non-trivial algebraic relation $f([\text{Sym}^1 X], \dots, [\text{Sym}^j X]) = 0$ for some j , contradicting Lemma 3.19. \square

3.23. Proof of Theorem 1.13. We finally prove Theorem 1.13 using Proposition 3.12 and Lemma 3.18.

From (3.11), $\sum_{s \geq 0} \mathbf{Z}_{X, *^s}^{-1}(t) = 1$: consider the contribution from the right side of (3.11) to the term w_v for each v , and note that if $v \neq \emptyset$, say $v = *^k \pi$, then w_v has contributions from the $s = 0$ term and the $s = k$ term, with opposite signs.

Thus, multiplying (3.13) by $Z_X(t)$, and summing over $\lambda = *^s$ for $s \geq 0$, we obtain

$$\sum_{\lambda \in \mathcal{P}} w_\lambda t^{\Sigma \lambda} = Z_X(t) = \sum_{s \geq 0} Z_{X,*^s}^{-1}(t) Z_X(t) = \sum_{\pi \in \mathcal{P}} c_{\pi,*^s} w_\pi.$$

Since Lemma 3.18 implies that the linear relations among the w_λ with fixed $|\lambda|$ generate all the linear relations among the w_λ , we can deduce that

$$(3.24) \quad Z_X^s(t) = \sum_{\substack{\lambda \in \mathcal{P} \\ |\lambda|=s}} w_\lambda t^{\Sigma \lambda} = \sum_{\substack{\pi \in \mathcal{P} \\ |\pi|=s}} c_{\pi,*^s} w_\pi = Z_{X,*^s}^{-1}(t) Z_X(t).$$

Thus for $j \geq j_s$,

$$\begin{aligned} \frac{W_{*^s}}{\mathbf{L}^r} &\equiv \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} \frac{w_{\mu *^s}}{\mathbf{L}^{|\mu *^s|(d+1)}} \pmod{\text{codim} > s} \quad (\text{Prop. 3.6}) \\ &= Z_{X,*^s}^{-1}(\mathbf{L}^{-(d+1)}) \quad (\text{from (3.11)}) \\ &= \left(\sum_{|\lambda|=s} \frac{w_\lambda}{\mathbf{L}^{(d+1)\Sigma \lambda}} \right) \frac{1}{Z_X(\mathbf{L}^{-(d+1)})} \quad (\text{from (3.24)}), \end{aligned}$$

which proves Theorem 1.13.

3.25. *Remark: s ordered points (cf. Remark 1.15(vii)).* Let λ be the partition $1^2 \cdots s$. Then

$$\frac{[H^0(X, \mathcal{L}^{\otimes j})^{s \text{ ordered}}]}{[H^0(X, \mathcal{L}^{\otimes j})]} = \frac{[W_\lambda]}{\mathbf{L}^r}$$

by the definition of W_λ (where $r = h^0(X, \mathcal{L}^{\otimes j})$) as in Lemma 3.2). Then by Proposition 3.6,

$$\lim_{j \rightarrow \infty} \frac{[H^0(X, \mathcal{L}^{\otimes j})^{s \text{ ordered}}]}{[H^0(X, \mathcal{L}^{\otimes j})]} = \sum_{\mu \in \mathcal{Q}} (-1)^{\|\mu\|} \frac{[w_{\lambda \mu}]}{\mathbf{L}^{|\lambda \mu|(d+1)}},$$

which is $Z_{X,\lambda}^{-1}(\mathbf{L}^{-d-1})$ by the definition (3.11) of $Z_{X,\lambda}^{-1}(t)$. By inserting Lemma 3.16 into (3.17), after slight rearranging we have

$$Z_{X,\lambda}^{-1}(t) = [w_\lambda] t^{|\lambda|} Z_X^{-1}(t) - \sum_{k=1}^s \binom{s}{k} (-1)^k t^k Z_{X,\lambda}^{-1}(t)$$

from which $Z_{X,\lambda}^{-1}(t) = \frac{w_\lambda t^s Z_X^{-1}(t)}{(1-t)^s}$. Remark 1.15(vii) follows.

4. MOTIVIC STABILIZATION OF SYMMETRIC POWERS

In this section, we prove three statements given in the introduction.

4.1. Proposition. — *Suppose X is a geometrically irreducible variety. Then $[\text{Sym}^n(X \times \mathbb{A}^1)] = \mathbf{L}^n \times [\text{Sym}^n X]$, and hence motivic stabilization of symmetric powers holds for X if and only if it holds for $X \times \mathbb{A}^1$.*

This is essentially [GZLMH1, Statement 3], and also follows by applying Totaro's argument of [G2, Lemma 4.4]. The same argument applies for MSSP_ϕ for any motivic measure ϕ .

4.2. Proposition. — *Suppose X is a geometrically irreducible variety, $U \subset X$ is a dense open set, and $Y \subset X$ is the complementary closed set. Then*

$$(4.3) \quad \lim_{n \rightarrow \infty} \frac{[\text{Sym}^n X]}{\mathbf{L}^{dn}} = \mathbf{Z}_Y(\mathbf{L}^{-d}) \lim_{n \rightarrow \infty} \frac{[\text{Sym}^n U]}{\mathbf{L}^{dn}}.$$

More precisely the limit on the left exists (motivic stabilization of symmetric powers holds for X) if and only if the limit on the right exists (motivic stabilization of symmetric powers holds for U), and in this case (4.3) holds. In particular, if X_1 and X_2 are birational geometrically irreducible varieties, then motivic stabilization of symmetric powers holds for X_1 if and only if it holds for X_2 .

Proof. We prove the result modulo dimension $-k$ classes, by induction on k . The case $k = 0$ is trivial. For all n ,

$$(4.4) \quad \frac{[\text{Sym}^n X]}{\mathbf{L}^{dn}} = \frac{[\text{Sym}^n U]}{\mathbf{L}^{dn}} + \frac{[\text{Sym}^{n-1} U]}{\mathbf{L}^{d(n-1)}} \frac{[\text{Sym}^1 Y]}{\mathbf{L}^d} + \cdots + \frac{[\text{Sym}^{n-k} U]}{\mathbf{L}^{d(n-k)}} \frac{[\text{Sym}^k Y]}{\mathbf{L}^{dk}}$$

modulo classes of dimension less than $-k$. (Here we use that $\dim[\text{Sym}^m Y]/\mathbf{L}^{dm} \leq -m$, and $\dim[\text{Sym}^m U]/\mathbf{L}^{dm} = 0$.) If the symmetric powers of U motivically stabilize, to $\text{SP}(U)$ say, then the right side stabilizes to $\text{SP}(U)\mathbf{Z}_Y(\mathbf{L}^{-d})$ modulo classes of dimension $< -k$, as desired. On the other hand, if the symmetric powers of X stabilize, and the symmetric powers of U stabilize up to dimension $-k + 1$, then everything in (4.4) stabilizes (modulo classes of dimension $< -k$) except for possibly $[\text{Sym}^n U]/\mathbf{L}^{dn}$; but then this class must stabilize as well, as desired. \square

4.5. Proposition. — *If X is a geometrically irreducible smooth projective curve with a rational point, then motivic stabilization of symmetric powers holds for X .*

Proof. For $n > 2g - 2$ (where g is the genus of X), $\text{Sym}^n X$ is a (Zariski) \mathbb{P}^{n-g} -bundle over $\text{Jac } C$ (where we use the point to determine an isomorphism $\text{Pic}^n X \cong \text{Jac } C$, and so that $\text{Sym}^n X$ is a Zariski bundle), so $[\text{Sym}^n X] = [\mathbb{P}^{n-g}][\text{Jac } C]$. \square

5. MODULI OF POINTS: CONFIGURATION SPACES

Throughout this section, X will be a geometrically irreducible variety of dimension d . For notational convenience, define $\mathbf{M} = \mathbf{L}^d$. In this section, we will show that (under appropriate motivic stabilization hypotheses) the classes of all “discriminants” “stabilize” (as the number of points tends to ∞) to finite formulas in terms of motivic zeta values, which can be interpreted in terms of “motivic probabilities” (Theorem 1.30, see Corollary 5.7 below). Corollary 5.7 is a consequence of an unconditional statement about generating functions (Theorem 5.2), which is the most complicated result in this section.

Explicit special cases can be shown without the full strength of Theorem 5.2, and are sprinkled throughout.

We first name the generating functions that are the subject of our investigation. For each partition of positive integers ν , define

$$K_{1 \bullet \nu}(t) := \sum_j w_{1j\nu}(X)t^j \quad \text{and} \quad \bar{K}_{1 \bullet \nu}(t) := \sum_j \bar{w}_{1j\nu}(X)t^j.$$

In the proofs of our results, we use a generalization of $K_{1 \bullet \nu}(t)$. For an integer a , define $w_{(<a \vdash j)\nu}(X) := \sum w_{\mu\nu}(X)$, where the sum is over the set of partitions μ of j with all parts less than a . Define

$$(5.1) \quad K_{(<a)\nu} := \sum_j w_{(<a \vdash j)\nu}(X)t^j,$$

so $K_{(<2)\nu} = K_{1 \bullet \nu}$. Informally: $(< a \vdash j)$ refers the set of partitions of j with parts $< a$, and $(< a)$ refers to the set of all partitions with all parts $< a$.

5.2. Theorem. — *For each partition ν of positive integers and each $a \geq 2$, there exist universal formulas $A_{\nu,a}(t)$, $B_{\nu,a}(t)$, $C_\nu(t)$, and $D_\nu(t)$ (recursively defined in Propositions 5.11 and 5.14(b)) such that*

- $A_{\nu,a}(t)$ is a $\mathbb{Z}[t]$ -linear combination of $w_{\nu'}$, where $|\nu'| = |\nu|$ and $m(\nu') \leq m(\nu)$;
- $C_\nu(t)$ is a $\mathbb{Z}[t]$ -linear combination of $w_{\nu'}/Z_X(t^i)$, where $|\nu'| \leq |\nu| - 1$, $i \in \nu$, and $i \geq 2$;
- and $B_{\nu,a}(t)$, $D_\nu(t) \in \mathbb{Z}[t]$, both having constant coefficient 1;

such that for any X and \mathbb{K} ,

(a) *when ν has all parts at least a ,*

$$K_{(<a)\nu}(t) = \frac{Z_X(t)}{Z_X(t^a)} \frac{A_{\nu,a}(t)}{B_{\nu,a}(t)},$$

(b)

$$\bar{K}_{1 \bullet \nu}(t) = Z_X(t) \frac{C_\nu(t)}{D_\nu(t)}.$$

These formulas are independent of X and \mathbb{K} , in the sense that their only dependence on X is via the universal formulas (in terms of the $\text{Sym}^\bullet X$) for $w_{\nu'}(X)$ (given in §3.20) and $Z_X(t)$. Because the formulas for $A_{\nu,a}(t)$ through $D_\nu(t)$ are finite, the formulas for $K_{(<a)\nu}(t)$ and $\bar{K}_{1 \bullet \nu}(t)$ have finite descriptions in terms of zeta functions. Part (a) will follow from Proposition 5.11, and part (b) will follow from part (a) and 5.14(b) (see the comment after the statement of Proposition 5.14). Before embarking on the proof, we give some interpretations and special cases.

5.3. Interpretations and consequences.

By the following lemma, the formulas for $K_{(<a)v}(t)$ (hence $K_{1\bullet v}(t)$), and $\bar{K}_{1\bullet v}(t)$ imply similar formulas for the limits of their coefficients.

5.4. Lemma. — Suppose $Y(t) = E(t)Z_X(t)$, where $Y(t) = \sum_j Y_j t^j$ and $E(t) = \sum_j E_j t^j$ both lie in $\mathcal{M}_L[[t]]$.

(a) If X satisfies $MSSP_\phi$, and $E(\mathbf{M}^{-1})$ exists (i.e., converges in $\phi(\widehat{\mathcal{M}}_L)$), then

$$(5.5) \quad \lim_{j \rightarrow \infty} \frac{Y_j}{\mathbf{M}^j} = E(\mathbf{M}^{-1})SP_\phi(X) \quad \text{in } \phi(\widehat{\mathcal{M}}_L).$$

If further the $[\text{Sym}^j X]$ are invertible in $\phi(\widehat{\mathcal{M}}_L)$ (e.g. if X is rational or $\phi = \text{HS}$, §1.5), then

$$(5.6) \quad \lim_{j \rightarrow \infty} \frac{Y_j}{[\text{Sym}^j X]} = E(\mathbf{M}^{-1}) \quad \text{in } \phi(\widehat{\mathcal{M}}_L).$$

(b) If $\mathbb{K} = \mathbb{F}_q$, and $\#E(1/q^d)$ converges (in \mathbb{R}), then

$$\lim_{j \rightarrow \infty} \frac{\#Y_j}{\#\text{Sym}^j X} = \#E(1/q^d).$$

The stabilization of the symmetric powers of X in the motivic measure ϕ , that is, $SP_\phi(X) = \lim_{n \rightarrow \infty} \frac{[\text{Sym}^n X]}{\mathbf{L}^{dn}}$, was defined in §1.24. Part (b) may be interpreted as inspiration for (a), but does not follow from (a), as the point-counting map $\#$ does not extend to a map $\widehat{\mathcal{M}}_L \rightarrow \mathbb{R}$ (see §1.4).

Proof. (a) We prove (5.5) in the case where ϕ is the identity, i.e. when motivic stabilization holds in general. The extension to a particular ϕ is straightforward using the continuity of ϕ , but the argument is notationally more cumbersome, hence left to the reader.

To prove the result, we prove the result modulo dimension at most $-b$, for any b . By hypothesis $E_i \mathbf{M}^{-i}$ is bounded above in dimension by some integer b_1 . Let b_2 be such that if $i \geq b_2$, then $E_i \mathbf{M}^{-i}$ has dimension at most $-b$, and b_3 be such that for $i \geq b_3$, we have that $SP(X) - [\text{Sym}^i X] \mathbf{M}^{-i}$ has dimension at most $-b - b_1$. For $j \geq b_2 + b_3$ we have, modulo dimension at most $-b$,

$$\frac{Y_j}{\mathbf{M}^j} = \sum_{i=0}^j \frac{E_{j-i}}{\mathbf{M}^{j-i}} \frac{[\text{Sym}^i X]}{\mathbf{M}^i} \equiv \sum_{i=j-b_2}^j \frac{E_{j-i}}{\mathbf{M}^{j-i}} \frac{[\text{Sym}^i X]}{\mathbf{M}^i} \equiv \sum_{i=j-b_2}^j \frac{E_{j-i}}{\mathbf{M}^{j-i}} SP(X) \equiv \sum_{i=0}^\infty \frac{E_i}{\mathbf{M}^i} SP(X).$$

Part (b) is an exercise in convergent power series, using the fact (from the Weil conjectures) that the unique root of the denominator of $\zeta_X(t)$ with the smallest absolute value is $1/q^d$ (cf. Motivation 1.26(i)). \square

Combining Theorem 5.2 (still to be proved) and Lemma 5.4, we have the limits of normalized configuration spaces promised in the introduction (Theorem 1.30). The hypotheses of Lemma 5.4 are straightforward to check.

5.7. Corollary. — Suppose v is a partition of positive integers, all at least 2.

(a) If X satisfies $MSSP_\phi$, then

$$\lim_{j \rightarrow \infty} \frac{w_{1^j v}}{\mathbf{M}^j} = \frac{1}{\zeta_X(2d)} \frac{A_{v,2}(\mathbf{M}^{-1})}{B_v(\mathbf{M}^{-1})} SP_\phi(X) \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{\overline{w}_{1^j v}}{\mathbf{M}^j} = \frac{C_{v,2}(\mathbf{M}^{-1})}{D_v(\mathbf{M}^{-1})} SP_\phi(X)$$

in $\phi(\widehat{\mathcal{M}}_L)$. If furthermore the $[\text{Sym}^j X]$ are invertible (e.g. if X is rational or $\phi = \text{HS}$, §1.5), then

$$\lim_{j \rightarrow \infty} \frac{[w_{1^j v}]}{[\text{Sym}^{j+|v|} X]} = \frac{1}{\mathbf{M}^{|v|} \zeta_X(2d)} \frac{A_{v,2}(\mathbf{M}^{-1})}{B_v(\mathbf{M}^{-1})} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{[\overline{w}_{1^j v}]}{[\text{Sym}^{j+|v|} X]} = \frac{1}{\mathbf{M}^{|v|}} \frac{C_{v,2}(\mathbf{M}^{-1})}{D_v(\mathbf{M}^{-1})}.$$

(b) If $\mathbb{K} = \mathbb{F}_q$, then

$$\lim_{j \rightarrow \infty} \frac{\#w_{1^j v}}{\#\text{Sym}^{j+|v|} X} = \frac{1}{q^{d|v|} \zeta_X(2d)} \frac{\#A_{v,2}(q^{-d})}{\#B_v(q^{-d})} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{\#\overline{w}_{1^j v}}{\#\text{Sym}^{j+|v|} X} = \frac{1}{q^{d|v|}} \frac{\#C_{v,2}(q^{-d})}{\#D_v(q^{-d})}.$$

5.8. Determining the universal formulas $A_{v,a}(t)$ through $D_v(t)$.

We now begin the proof of Theorem 5.2. En route, we in effect prove special cases, giving explicit descriptions of $A_{v,a}(t)$ through $D_v(t)$. We start with an important base case.

5.9. Proposition (see §1.35). — If $a > 1$,

- (a) $K_{(<a)}(t) = Z_X(t)/Z_X(t^a)$.
- (b) $\overline{K}_{1^\bullet(a)}(t) = t^{-a} Z_X(t)(1 - 1/Z_X(t^a))$.

Proof. We have

$$[\text{Sym}^j X] t^j = \sum_{\lambda \vdash j} [w_\lambda] t^{\sum \lambda} = \sum_{\mu \in \mathcal{P}} [w_{a \times \mu}] t^{a \sum \mu} [w_{(<a \vdash j-a \sum \mu)}] t^{j-a \sum \mu},$$

where the notation $a \times p$ (used only in this proof) denotes the partition obtained by multiplying all of the elements of p by a . (Given λ , to find the μ on the right side, “round down” the parts of λ to the next multiple of a , then divide the result by a . A similar idea, with $a = 2$, is used in the proof of Theorem 5.10 below.) Using $[w_{a \times \mu}] = [w_\mu]$, we have

$$\begin{aligned} Z_X(t) &= \sum_j [\text{Sym}^j X] t^j \\ &= \sum_j \sum_{\mu \in \mathcal{P}} [w_{a \times \mu}] t^{a \sum \mu} [w_{(<a \vdash j-a \sum \mu)}] t^{j-a \sum \mu} \\ &= \left(\sum_{\mu \in \mathcal{P}} [w_\mu] t^{a \sum \mu} \right) \left(\sum_k [w_{(<a \vdash k)}] t^k \right) \\ &= Z_X(t^a) K_{(<a)}(t), \end{aligned}$$

yielding part (a). Part (b) then follows from (a) via the identity $K_{(<a)}(t) + t^a \overline{K}_{\bullet(a)}(t) = Z_X(t)$: each partition of n either has all parts smaller than a , or else at least one part at least a . \square

5.10. Theorem. —

$$\sum_j [\text{Sym}_s^j X] t^j = \frac{Z_X^{[s]}(t^2) Z_X(t)}{Z_X(t^2)}.$$

Theorem 1.39 follows by combining Theorem 5.10 and Lemma 5.4.

Proof. Let \mathcal{S}_s be the set of partitions of positive integers λ with $|\lambda| = s$ and with all parts even. For a partition λ with all parts even, let $\mathcal{T}_{\lambda,j}$ be the set of all partitions $\mu \vdash j$ such that $\{2\lfloor \mu_i/2 \rfloor \mid 2\lfloor \mu_i/2 \rfloor > 0\} = \lambda$, (i.e. λ is obtained from μ by rounding the parts down to the nearest even integer and discarding 0's, cf. the proof of Proposition 5.9(a)). (The notation \mathcal{S}_s and $\mathcal{T}_{\lambda,j}$ will only be used in this proof.)

For a partition λ with all parts even, $w_\lambda w_{1j-\Sigma \lambda} = \sum_{\mu \in \mathcal{T}_{\lambda,j}} w_\mu$, so

$$\sum_{\lambda \in \mathcal{S}_s} [w_\lambda] [w_{1j-\Sigma \lambda}] = \sum_{\lambda \in \mathcal{S}_s} \sum_{\mu \in \mathcal{T}_{\lambda,j}} [w_\mu] = [\text{Sym}_s^j X],$$

as the middle double sum enumerates the partitions μ of j with precisely s multiple points. Thus

$$\sum_j [\text{Sym}_s^j X] t^j = \sum_j \sum_{\lambda \in \mathcal{S}_s} [w_\lambda] [w_{1j-\Sigma \lambda}] t^j = \left(\sum_{\lambda \in \mathcal{S}_s} [w_\lambda] t^{\Sigma \lambda} \right) \left(\sum_k [w_{1k}] t^k \right) = Z_X^{[s]}(t^2) \frac{Z_X(t)}{Z_X(t^2)}.$$

where the last equality uses Proposition 5.9(a). \square

Temporarily (for the purpose of Proposition 5.11) define $\mathcal{A}_{<a}(\nu)$ as the set of all partitions obtained by adding an element of $\{0, \dots, a-1\}$ to each of the parts of ν . (Think: “Add $< a$ to the parts of ν ”.) For example, $[x+2, x+2, x, y+1, y] \in \mathcal{A}_{<a}([x, x, x, y, y])$ for $a \geq 3$.

5.11. Proposition (recursion for $K_{(<a)\nu}(t)$). — *For any formalization ν of a partition,*

$$K_{(<a)\nu}(t) = \frac{\frac{Z_X(t)}{Z_X(t^a)} w_\nu - \sum_{\substack{\nu' \in \mathcal{A}_{<a}(\nu) \\ m(\nu') < m(\nu)}} K_{(<a)\nu'}(t) t^{\Sigma \nu' - \Sigma \nu}}{\sum_{\substack{\nu' \in \mathcal{A}_{<a}(\nu) \\ m(\nu') = m(\nu)}} t^{\Sigma \nu' - \Sigma \nu}}.$$

Note that the denominator is a polynomial with constant coefficient 1, as $\nu' = \nu$ appears in the bottom sum. Theorem 5.2(a) follows inductively from Proposition 5.11, using the base case $\nu = \emptyset$, because we may replace ν by its formalization. (The assumption that all the parts of ν are at least a in Theorem 5.2(a) arises because of this need to replace ν by its formalization.)

Proof. For any a and j ,

$$(5.12) \quad w_{(<a|+j)} w_v = \sum_{v' \in \mathcal{A}_{<a}(v)} w_{(<a|+j-\sum v'+\sum v)v'}.$$

Reason: when multiplying $w_{(<a|+j)}$ with w_v , the right side keeps track of “how the points parametrized by $w_{(<a|+j)}$ and w_v overlap”. Multiplying (5.12) by t^j and summing over all j , we have

$$\begin{aligned} \sum_j [w_{(<a|+j)}] [w_v] t^j &= \sum_j t^j \sum_{v' \in \mathcal{A}_{<a}(v)} [w_{(<a|+j-\sum v'+\sum v)v'}] \\ K_{(<a)}(t) [w_v] &= \sum_{v' \in \mathcal{A}_{<a}(v)} t^{\sum v'-\sum v} \sum_k t^k [w_{(<a|+k)v'}] \quad (\text{by (5.1)}) \\ \frac{Z_X(t)}{Z_X(t^a)} [w_v] &= \sum_{v' \in \mathcal{A}_{<a}(v)} t^{\sum v'-\sum v} K_{(<a)} v' \quad (\text{Prop. 5.9(a) and (5.1)}) \end{aligned}$$

A little thought shows that if v is the formalization of a partition, and $v' \in \mathcal{A}_{<a}(v)$, then $m(v') \leq m(v)$, and if furthermore $m(v') = m(v)$, then $K_{(<a)} v' = K_{(<a)} v$. Thus

$$\begin{aligned} \frac{Z_X(t)}{Z_X(t^a)} [w_v] &= \sum_{\substack{v' \in \mathcal{A}_{<a}(v) \\ m(v') < m(v)}} t^{\sum v'-\sum v} K_{(<a)} v' + \sum_{\substack{v' \in \mathcal{A}_{<a}(v) \\ m(v') = m(v)}} t^{\sum v'-\sum v} K_{(<a)} v' \\ &= \sum_{\substack{v' \in \mathcal{A}_{<a}(v) \\ m(v') < m(v)}} t^{\sum v'-\sum v} K_{(<a)} v' + \left(\sum_{\substack{v' \in \mathcal{A}_{<a}(v) \\ m(v') = m(v)}} t^{\sum v'-\sum v} \right) K_{(<a)} v \end{aligned}$$

The result follows. \square

5.13. *Example (see §1.37).* If v has all distinct elements greater than 1, then Proposition 5.11 inductively yields

$$K_{1 \bullet v}(t) = \frac{Z_X(t)}{Z_X(t^2)} \frac{w_v}{(1+t)^{|v|}}.$$

Temporarily (for the purpose of Proposition 5.14) define $\mathcal{S}(v, a)$ to be the (finite) set of partitions μ with all parts at least a such that there exists a partition π with elements each $< a$ with $1^{|v|(a-1)} v \leq \pi \mu \not\leq 1^{|v|(a-1)-a} a v$. In other words, these are the partitions (up to “small parts” $< a$) which can be obtained by merging in $|v|(a-1)$ ones with v , but which cannot be obtained if a of the ones are merged together first.

5.14. Proposition. —

(a) For a partition v of positive integers, and an integer a no bigger than the smallest part of v ,

$$[\overline{w}_{1^{j-a} a v}] = [\overline{w}_{1^j v}] - \sum_{\mu \in \mathcal{S}(v, a)} [w_{(<a|+j-\sum \mu+\sum v)\mu}].$$

(b) For any partition ν of positive integers all parts at least a ,

$$\overline{K}_{1^\bullet a\nu}(t) = \overline{K}_{1^\bullet \nu}(t)t^{-a} - \sum_{\mu \in S(\nu, a)} K_{(<a)\mu}(t)t^{-a+\sum \mu - \sum \nu}.$$

Theorem 5.2(b) follows from inductively from Propositions 5.14(b) and Theorem 5.2(a), and the base case $\overline{K}_{1^\bullet \emptyset}(t) = \mathbf{Z}_X(t)$.

Proof. (a) By considering which w_λ are contained in $\overline{w}_{1^j \nu}$ and in $\overline{w}_{1^{j-a} a\nu}$, we have

$$(5.15) \quad [\overline{w}_{1^j \nu}] - [\overline{w}_{1^{j-a} a\nu}] = \sum_{1^j \nu \leq \lambda \not\leq 1^{j-a} a\nu} [w_\lambda].$$

We give a name to the partitions appearing on the right side of (5.15): let $\mathcal{T}(j) := \{\lambda \mid 1^j \nu \leq \lambda \not\leq 1^{j-a} a\nu\}$. For each $\lambda \in \mathcal{T}(j)$, we write $\lambda = b(\lambda)s(\lambda)$, where the “big” part $b(\lambda)$ is a partition composed of the elements of λ that are $\geq a$ and the “small” part $s(\lambda)$ is the rest. Note that in any merge that created λ from $1^j \nu$, only 1’s can contribute to the $s(\lambda)$ part.

Note that if all elements of an integer partition μ are at least a , then (if $\sum 1^j \mu = \sum \lambda$) whether $1^j \mu \leq \lambda$ depends only on $b(\lambda)$. In particular, if $\mu = b(\lambda)$ for some $\lambda \in \mathcal{T}(j)$, then for all partitions π of $j - \sum \mu + \sum \nu$ into elements less than a , we have that $\mu\pi \in \mathcal{T}(j)$.

Also note that

$$(5.16) \quad \{\mu \mid \mu = b(\lambda) \text{ for some } \lambda \in \mathcal{T}(j)\} = \{\mu \mid \mu = b(\lambda) \text{ for some } \lambda \in \mathcal{T}(|\nu|(a-1))\} \\ \text{and } \sum \mu - \sum \nu \leq j\}.$$

Thus

$$\begin{aligned} [\overline{w}_{1^j \nu}] - [\overline{w}_{1^{j-a} a\nu}] &= \sum_{\lambda \in \mathcal{T}(j)} [w_\lambda] \quad (\text{by definition of } \mathcal{T}(j)) \\ &= \sum_{\mu \in S(\nu, a)} [w_{(<a+j-\sum \mu + \sum \nu)\mu}], \end{aligned}$$

where of course $[w_{(<a+j-\sum \mu + \sum \nu)\mu}] = 0$ if $j - \sum \mu + \sum \nu < 0$.

(b) Multiply both sides of (a) by t^{j-a} , and sum over all j . □

5.17. Another example. Proposition 5.14(b) gives a recursion to compute $\overline{K}_{1^\bullet \nu}$ in all cases, but for some ν we have more efficient formulas. One example is Proposition 5.9(b) above. Another is the following Lemma, which provoked Conjecture E (§ 1.43), a topological conjecture about Betti numbers.

5.18. Lemma. — For $1 < a \leq b$ and $r \geq 0$,

$$\overline{w}_{1^{j-a} a b^r} = \overline{w}_{1^j b^r} - \overline{w}_{x^j y^r} + \overline{w}_{x^{j-a} (ax)^r y^r},$$

where x and y are formal variables.

(Caution: a and b are integers, while x and y are formal variables — this is key to the argument!)

Proof. If μ is a partition, let \mathcal{R}_μ be the set of partitions $\geq \mu$, i.e. obtainable from μ by merging. We have a map of posets $\mathcal{R}_{x^j y^r} \rightarrow \mathcal{R}_{1^j b^r}$ sending $x \mapsto 1$ and $y \mapsto b$. We claim that the map $\mathcal{R}_{x^j y^r} \rightarrow \mathcal{R}_{1^j b^r}$ restricts to a *bijection* $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r} \rightarrow \mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}$, and that this bijection preserves the multiplicity sequence of each partition.

First, we will see that $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r}$ does map to $\mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}$. Consider an element μ of $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r}$ that maps to λ in $\mathcal{R}_{1^j b^r}$. Each element of μ has at most $(a-1)$ x 's, and thus the reduction of an element of λ modulo b is between 0 and $a-1$. In particular, the sum of these reductions (as integers, not modulo b) is j . If λ were in $\mathcal{R}_{1^{j-a}ab^r}$, it would either have an element whose reduction is between a and $b-1$ modulo b , or the sum of the reductions modulo b of the elements of λ would be less than j .

Second, given $\lambda \in \mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}$, looking at the residues of the elements modulo b , we know where all the 1's have gone in any merge, and thus where all the b 's are, determining a pre-image on $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r}$ uniquely. Finally, since no element of $\lambda \in \mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r}$ has more than $(a-1)$ x 's, if two elements $c_1 x + c_2 y = c_3 x + c_4 y$ ($c_1, \dots, c_4 \in \mathbb{Z}^{\geq 0}$) are equal after the map to $\mathcal{R}_{1^j b^r}$, then we have

$$c_1 + c_2 b = c_3 + c_4 b$$

for $0 \leq c_1, c_3 \leq a-1$, and thus $c_1 = c_3$ and $c_2 = c_4$. This shows that the map $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r} \rightarrow \mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}$ preserves multiplicity sequences.

The bijection $\mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r} \rightarrow \mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}$ thus gives

$$\overline{w}_{1^j b^r} - \overline{w}_{1^{j-a}ab^r} = \sum_{\lambda \in \mathcal{R}_{1^j b^r} \setminus \mathcal{R}_{1^{j-a}ab^r}} w_\lambda = \sum_{\mu \in \mathcal{R}_{x^j y^r} \setminus \mathcal{R}_{x^{j-a}(ax)y^r}} w_\mu = \overline{w}_{x^j y^r} - \overline{w}_{x^{j-a}(ax)y^r}.$$

□

5.19. Proposition (see §1.36 and Conjecture 1.43). — *Given $1 < a \leq b$ and $r \geq 0$, we have*

$$\overline{K}_{1 \bullet ab^r}(t) = \overline{K}_{1 \bullet b^r}(t) t^{-a} - \frac{Z_X(t) t^{-a}}{Z_X(t^a)} [\text{Sym}^r X].$$

Proof. Multiplying Lemma 5.18 by t^{j-a} and summing over j , we obtain

$$\begin{aligned} \overline{K}_{1 \bullet ab^r}(t) &= \overline{K}_{1 \bullet b^r}(t) t^{-a} - \sum_j [\text{Sym}^j X] ([\text{Sym}^r X] t^{j-a}) + \sum_j \overline{w}_{x_1^{j-a}(ax)_1} t^{j-a} \overline{w}_{x_1^r} \\ &= \overline{K}_{1 \bullet b^r}(t) t^{-a} - Z_X(t) [\text{Sym}^r X] t^{-a} + \sum_j ([\text{Sym}^j X] - [w_{(<a, j)}]) t^{j-a} [\text{Sym}^r X] \\ &= \overline{K}_{1 \bullet b^r}(t) t^{-a} - Z_X(t) [\text{Sym}^r X] t^{-a} + (Z_X(t) - K_{(<a)}(t)) [\text{Sym}^r X] t^{-a}. \end{aligned}$$

The result then follows from Proposition 5.9(a). □

5.20. *Example (see §1.36 and Conjecture 1.43).* For $1 < a \leq b$ and $r \geq 0$, we have

$$\bar{K}_{1 \bullet_{ab} r}(t) = t^{-a-rb} \left(Z_X(t) - \frac{Z_X(t)}{Z_X(t^b)} \left(\sum_{i=0}^{r-1} [\text{Sym}^i X] t^{bi} \right) - \frac{Z_X(t)}{Z_X(t^a)} [\text{Sym}^r X] t^{rb} \right),$$

by applying Proposition 5.19 inductively. (Note that $\bar{K}_{1 \bullet_{br}}(t)$ should be interpreted as $\bar{K}_{1 \bullet_{bb^{r-1}}}(t)$ to be computed inductively.) For example, if $X = \mathbb{A}^d$, then $Z_X(t) = 1/(1 - Mt)$ (see the start of §1.42) and $\bar{K}_{1 \bullet_{ab} r}(t) = M^{r+1}/(1 - Mt)$.

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